Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems

Guy Métivier
Université Bordeaux 1, IMB UMR CNRS 5251
33405 Talence Cedex, France
guy.metivier@math.u-bordeaux1.fr

May 9, 2008

Contents

Preface

These notes originate from a graduate course given at the University of Pisa during the spring semester 2007. They were completed while the author was visiting the Centro di Ricerca Matematica Ennio De Giorgi in february 2008. The author thanks both institutions for their warm hospitality.

The main objective is to present at the level of beginners an introduction to several modern tools of micro-local analysis which are useful for the mathematical study of nonlinear partial differential equations. The guideline is to show how one can use the para-differential calculus to prove energy estimates using para-differential symmetrizers, or to decouple and reduce systems to equations. In these notes, we have concentrated the applications on the well posed-ness of the Cauchy problem for nonlinear PDE's. It is important to note that the methods presented here do apply to other problems, such as, elliptic equations, propagation of singularities (see the original article of J-M Bony [Bon]), boundary value problems, shocks, boundary layers (see e.g [Mé1, Mé2, MZ]). In particular, in applications to physical problems, the use of para-differential symmetrizers for boundary value problems is much more relevant for hyperbolic boundary value problems than for the hyperbolic Cauchy problem where there are more direct estimates, relying on symmetry properties that are satisfied by many physical systems. However, the analysis of boundary value problems involve much more technicalities which we wanted to avoid in these introductory lectures. The Cauchy problem is a good warm up to become familiar with the technics.

These notes are divided in three parts. Part I is an introduction to evolution equations. After the presentation of physical examples, we give the bases of the analysis of systems with constant coefficients. The Fourier analysis provides both explicit solutions and *an exact symbolic calculus* for Fourier multipliers, which can be used for diagonalizing systems or constructing symmetrizers. The key word is *hyperbolicity.* However, we have restricted the analysis to strongly hyperbolic systems, aiming at simplicity and avoiding the subtleties of weak hyperbolicity.

In Part II, we give an elementary and self-contained presentation of the para-differential calculus which was introduced by Jean-Michel Bony [Bon] in 1979. We start with the Littlewood-Paley or harmonic analysis of classical function spaces (Sobolev spaces and Hölder spaces). Next we say a few words about the general framework of the classical pseudo-differential calculus and prove Stein's theorem for operators of type $(1, 1)$. We go on introducing symbols with limited smoothness and their *para-differential quantization* as operators of type $(1, 1)$. A key idea from J-M.Bony is that one can replace nonlinear expressions, thus nonlinear equations, by para-differential expressions, to the price of error terms which are much smoother than the main terms (and thus presumed to be harmless in the derivation of estimates). These are the *para-linearization theorems* which in nature are a linearization of the equations. We end the second part, with the presentation of an *approximate symbolic calculus*, which links the calculus of operators to a calculus for their symbols. This calculus which generalizes the exact calculus of Fourier multipliers, is really what makes the theory efficient and useful.

Part III is devoted to two applications. First we study quasi-linear hyperbolic systems. As briefly mentioned in Chapter 1, this kind of systems is present in many areas of physics (acoutics, fluid mechanics, electromagntism, elasticity to cite a few). We prove the local well posedness of the Cauchy problem for quasi-linear hyperbolic systems which admit a frequency dependent symmetrizer. This class is more general than the class of systems which are symmetric-hyperbolic in the sense of Friedrichs; it also incorporates all hyperbolic systems of constant multiplicity. *The key idea* is simple and elementary :

- 1) one looks for symmetrizers (multipliers) which are para-differential operators, that means that one looks for symbols;

- 2) one uses the symbolic calculus to translate the desired properties of the symmetrizers as operators into properties of their symbols;

- 3) one determines the symbols of the symmetrizers satisfying these properties. At this level, the computation is very close to the constant coefficient analysis of Part I.

Though most (if not all) physical examples are symmetric hyperbolic in the sense of Friedrichs, it is important to experiment such methods on the simpler case of the Cauchy problem, before applying them to the more delicate, but similar, analysis of boundary value problems where they appear to be much more significant for a sharp analysis of the well posed-ness conditions.

The second application concerns the local in time well posedness of the Cauchy problem for systems of Schödinger equations, coupled though quasi-linear interactions. These systems arise in nonlinear optics: each equation models the dispersive propagation of the envelop of a high frequency beam, the coupling between the equations models the interaction between the beams and the coupling is actually nonlinear for intense beams such as laser beams. This models for instance the propagation of a beam in a medium which by nonlinear resonance create scattered and back-scattered waves which interact with the original wave (see e.g. [Blo, Boy, NM, CCM]). It turns out that the system so obtained is not necessarily symmetric so that the energy estimates are not obtained by simple and obvious integrations by part. Here the symbolic calculus helps to understand what is going on. We use the symbolic-paradifferential calculus to decouple the systems and reduce the analysis to scalar equations. At this stage, the para-differential calculus can also be used to treat cubic interactions. The stress here that the results we give in Chapter 8 are not optimal neither the most general concerning Schödinger equations, but they appear as direct applications of the calculus developed in Part II. The sharp results require further work (see e.g. [KPV] and the references therein).

Part I

Introduction to Systems

Chapter 1

Notations and Examples.

This introductory chapter is devoted to the presentation of several classical examples of systems which occur in mechanics or physics. From the notion of plane wave, we first present the very important notions of dispersion relation or characteristic determinant, and of polarization of waves which are of fundamental importance in physics. From the mathematical point of view, this yields to introduce the notion of symbol of an equation and to study its eigenvalues and eigenvector. We illustrate these notions on the examples.

1.1 First order systems

1.1.1 Notations

We consider $N \times N$ systems of first order equations

$$A_0(t,x,u)\partial_t u + \sum_{j=1}^{d} A_j(t,x,u)\partial_{x_j} u = F(t,x,u) \tag{1.1.1}$$

where $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ denote the time-space variables; the A_j are $N \times N$ matrices and F is a function with values in $\mathbb{R}^N$; they depend on the variables (t,x,u) varying in a subdomain of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N$.

The Cauchy problem consists in solving the equation (1.1.1) together with initial conditions

$$u_{|t=0} = h. \tag{1.1.2}$$

We will consider only the case of *noncharacteristic Cauchy problems*, which means that A_0 is invertible. The system is linear when the A_j do not depend on u and F is affine in u, i.e. of the form $F(t,x,u) = f(t,x) + E(t,x)u$.

A very important case for applications is the case of *systems of conservation laws*

$$\partial_t f_0(u) + \sum_{j=1}^{d} \partial_{x_j} f_j(u) = 0. \tag{1.1.3}$$

For smooth enough solutions, the chain rule can be applied and this system is equivalent to

$$A_0(u)\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_{x_j} u = 0 \tag{1.1.4}$$

with $A_j(u) = \nabla_u f_j(u)$.

Consider a solution u_0 and the equation for small variations $u = u_0 + \varepsilon v$. Expanding in power series in ε yields at first order *the linearized equations*:

$$A_0(t,x,u_0)\partial_t v + \sum_{j=1}^{d} A_j(t,x,u_0)\partial_{x_j} v + E(t,x)v = 0 \tag{1.1.5}$$

where

$$E(t,x,v) = (v\cdot\nabla_u A_0)\partial_t u_0 + \sum_{j=1}^{d}(v\cdot\nabla_u A_j)\partial_{x_j} u_0 - v\cdot\nabla_u F$$

and the gradients $\nabla_u A_j$ and $\nabla_u F$ are evaluated at $(t,x,u_0(t,x))$.

In particular, the linearized equations from (1.1.3) or (1.1.4) near a constant solution $u_0(t,x) = \underline{u}$ are the *constant coefficients equations*

$$A_0(\underline{u})\partial_t u + \sum_{j=1}^{d} A_j(\underline{u})\partial_{x_j} u = 0. \tag{1.1.6}$$

1.1.2 Plane waves

Consider a *linear constant coefficient* system:

$$Lu := A_0\partial_t u + \sum_{j=1}^{d} A_j\partial_{x_j} u + Eu = f \tag{1.1.7}$$

Particular solutions of the homogeneous equation $Lu = 0$ are plane waves:

$$u(t, x) = e^{it\tau + ix\cdot\xi} a \tag{1.1.8}$$

where (τ, ξ) satisfy the *dispersion relation* :

$$\det\big(i\tau A_0 + \sum_{j=1}^{d} i\xi_j A_j + E\big) = 0 \tag{1.1.9}$$

and the constant vector a satisfies the *polarization condition*

$$a \in \ker\big(i\tau A_0 + \sum_{j=1}^{d} i\xi_j A_j + E\big). \tag{1.1.10}$$

The matrix $i\tau A_0 + \sum_{j=1}^{d} i\xi_j A_j + E$ is called *the symbol* of L.

In many applications, the coefficients A_j and E are real and one is interested in real functions. In this case (1.1.8) is to be replaced by $u = \mathrm{Re}\,(e^{it\tau + ix\cdot\xi} a)$.

When A_0 is invertible, the equation (1.1.9) means that $-\tau$ is an eigenvalue of $\sum \xi_j A_0^{-1} A_j - iA_0^{-1} E$ and the polarization condition (1.1.10) means that a is an eigenvector.

In many applications and in particular in the analysis of the Cauchy problem, one is interested in real wave numbers $\xi \in \mathbb{R}^d$. The well posedness for $t \geq 0$ of the Cauchy problem (for instance in Sobolev spaces) depends on the existence or not of exponentially growing modes $e^{it\tau}$ as $|\xi| \to \infty$. This leads to the condition, called *weak hyperbolicity* that there is a constant C such that for all $\xi \in \mathbb{R}^d$, the roots in τ of the dispersion relation (1.1.9) satisfy $\mathrm{Im}\,\tau \geq -C$. These ideas are developed in Chapter 2.

The *high frequency regime* is when $|\xi| \gg |E|$ (assuming that the size of the coefficients A_j is ≈ 1). In this regime, a perturbation analysis can be performed and L can be seen as a perturbation of the homogeneous system $L_0 = A_0\partial_t + \sum A_j \partial_{x_j}$. This leads to the notions of *principal symbol* $i\tau A_0 + \sum_{j=1}^{d} i\xi_j A_j$ and of *characteristic equation*

$$\det\big(\tau A_0 + \sum_{j=1}^{d} \xi_j A_j\big) = 0. \tag{1.1.11}$$

Note that the principal symbol and the characteristic equation are homogeneous in ξ, so that their analysis can be reduced to the sphere $\{|\xi| = 1\}$. In particular, for an homogeneous system L_0 weak hyperbolicity means that for all $\xi \in \mathbb{R}^d$, the roots in τ of the dispersion relation (1.1.11) are real.

However, there are many applications which are not driven by the high frequency regime $|\xi| \gg |E|$ and where the relevant object is the in-homogeneous dispersion relation (1.1.9). For instance, this is important when one wants to model the dispersion of light.

1.1.3 The symbol

Linear constant coefficients equations play an important role. First, they provide examples and models. They also appear as linearized equations (see (1.1.6)). In the analysis of linear systems

$$Lu := A_0(t,x)\partial_t u + \sum_{j=1}^{d} A_j(t,x)\partial_{x_j} u + E(t,x)u, \tag{1.1.12}$$

and in particular of linearized equations (1.1.5), they also appear by freezing the coefficients at a point $(\underline{t}, \underline{x})$.

This leads to the important notions of *principal symbol* of the nonlinear equation (1.1.1)

$$L(t,x,u,\tau,\xi) := i\tau A_0(t,x,u) + \sum_{j=1}^{d} i\xi_j A_j(t,x,u), \tag{1.1.13}$$

and of *characteristic equation* :

$$\det\big(\tau A_0(t,x,u) + \sum_{j=1}^{d} \xi_j A_j(t,x,u)\big) = 0, \tag{1.1.14}$$

where the variables (t,x,u,τ,ξ) are seen as independent variables in the phase space $\mathbb{R}^{1+d} \times \mathbb{R}^N \times \mathbb{R}^{1+d}$.

An important idea developed in these lectures is that many properties of the linear equation (1.1.12) and of the nonlinear equation (1.1.1) can be seen on the principal symbol. In particular, the spectral properties of

$$\sum_{j=1}^{d} \xi_j A_0^{-1}(t,x,u) A_j(t,x,u).$$

are central to the analysis. Properties such as reality, semi-simplicity, multiplicity of the eigenvalues or smoothness of the eigenvalues and eigenprojectors, are crucial in the discussions.

1.2 Examples

1.2.1 Gas dynamics

General Euler's equations

The equations of gas dynamics link the density ρ, the pressure p, the velocity $v = (v_1, v_2, v_3)$ and the total energy per unit of volume and unit of mass E through the equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \partial_t(\rho v_j) + \operatorname{div}(\rho v v_j) + \partial_j p = 0 \quad 1 \leq j \leq 3 \\ \partial_t E + \operatorname{div}(\rho E v + p v) = 0 \end{cases} \tag{1.2.1}$$

Moreover, $E = e + |v|^2/2$ where e is the specific internal energy. The variables ρ, p and e are linked by a state law. For instance, e can be seen as a function of ρ and p and one can take $u = (\rho, v, p) \in \mathbb{R}^5$ as unknowns. The second law of thermodynamics introduces two other dependent variables, the entropy S and the temperature T so that one can express p, e and T as functions $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{T}$ of the variables (ρ, S), linked by the relation

$$d\mathcal{E} = \mathcal{T} dS + \frac{\mathcal{P}}{\rho^2} d\rho\,. \tag{1.2.2}$$

One can choose $u = (\rho, v, S)$ or $\widetilde{u} = (p, v, S)$ as unknowns. The equations read (for smooth solutions):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0 \\ \rho(\partial_t v_j + v \cdot \nabla v_j) + \partial_j p = 0 \quad 1 \leq j \leq 3 \\ \partial_t S + v \cdot \nabla S = 0 \end{cases} \tag{1.2.3}$$

with p given a a given function $\mathcal{P}$ of (ρ, S) or ρ function of (p, S).

Perfect gases. They satisfy the condition

$$\frac{p}{\rho} = RT, \tag{1.2.4}$$

where R is a constant. The second law of thermodynamics (1.2.2) implies that

$$d\mathcal{E} = \frac{\mathcal{P}}{R\rho} dS + \frac{\mathcal{P}}{\rho^2} d\rho$$

thus

$$\frac{\partial \mathcal{E}}{\partial S} = \frac{\mathcal{P}}{R\rho}, \quad \frac{\partial \mathcal{E}}{\partial \rho} = \frac{\mathcal{P}}{\rho^2} \qquad \text{and} \qquad \rho \frac{\partial \mathcal{E}}{\partial \rho} - R \frac{\partial \mathcal{E}}{\partial S} = 0$$

Therefore, the relation between e, ρ and S has the form

$$e = \mathcal{E}(\rho, S) = F\big(\rho\, e^{S/R}\big). \tag{1.2.5}$$

Thus the temperature $T = \mathcal{T}(\rho, S) = \frac{\partial}{\partial S}\mathcal{E} = G\big(\rho\, e^{S/R}\big)$ with $G(s) = \frac{s}{R}F'(s)$. This implies that T is a function of e:

$$T = \Psi(e) = \frac{1}{R}G\big(F^{-1}(e)\big). \tag{1.2.6}$$

A particular case of this relation is when Ψ is linear, meaning that e is proportional to T:

$$e = CT, \tag{1.2.7}$$

with C constant. In this case

$$\frac{1}{R}sF'(s) = CF(s), \qquad \text{thus} \qquad F(s) = \lambda s^{RC}.$$

This implies that $e$and p are linked to ρ and S by

$$e = \rho^{\gamma-1}e^{C(S-S_0)}, \qquad p = (\gamma-1)\rho^{\gamma}e^{C(S-S_0)} = (\gamma-1)\rho e, \tag{1.2.8}$$

with $\gamma = 1 + RC$.

The symbol

The symbol of (1.2.3) is

$$i(\tau + v\cdot\xi)\mathrm{Id} + i\begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ c^2\xi_1 & 0 & 0 & 0 & 0 \\ c^2\xi_2 & 0 & 0 & 0 & 0 \\ c^2\xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.9}$$

where $c^2 := \dfrac{d\mathcal{P}}{d\rho}(\rho, S)$. The system is hyperbolic when $c^2 \geq 0$. For $\xi \neq 0$, the eigenvalues and eigenspaces are

$$\tau = -v\cdot\xi, \qquad \mathbb{E}_0 = \big\{\dot\rho = 0,\ \dot v \in \xi^{\perp}\big\}, \tag{1.2.10}$$

$$\tau = -v\cdot\xi \pm c|\xi|, \qquad \mathbb{E}_{\pm} = \Big\{\,\dot v = \pm c^2\dot\rho\frac{\xi}{|\xi|},\ \dot S = 0\Big\}. \tag{1.2.11}$$

The isentropic system

When S is constant the system (1.2.3) reduces to

$$\left\{\begin{array}{l}\partial_t\rho+\mathrm{div}(\rho v)=0\\ \rho(\partial_t v_j+v\cdot\nabla v_j)+\partial_j p=0\quad 1\leq j\leq 3\end{array}\right. \tag{1.2.12}$$

with ρ and p linked by a state law, $p=\mathcal{P}(\rho)$. For instance, $p=c\rho^\gamma$ for perfect gases satisfying (1.2.8).

Acoustics

By linearization of (1.2.12) around a constant state $(\underline{\rho},\underline{v})$, one obtains the equations

$$\left\{\begin{array}{l}(\partial_t+\underline{v}\cdot\nabla)\rho+\underline{\rho}\,\mathrm{div}v=f\\ \underline{\rho}(\partial_t+\underline{v}\cdot\nabla)v_j+\underline{c}^2\partial_j p=g_j\quad 1\leq j\leq 3\end{array}\right. \tag{1.2.13}$$

where $\underline{c}^2:=\dfrac{d\mathcal{P}}{d\rho}(\underline{\rho})$. Changing variables x to $x-t\underline{v}$, reduces to

$$\left\{\begin{array}{l}\partial_t\rho+\underline{\rho}\,\mathrm{div}v=f\\ \underline{\rho}\partial_t v+\underline{c}^2\nabla p=g.\end{array}\right. \tag{1.2.14}$$

1.2.2 Maxwell's equations

General equations

The general Maxwell's equations read:

$$\left\{\begin{array}{l}\partial_t D-c\,\mathrm{curl}H=-j,\\ \partial_t B+c\,\mathrm{curl}E=0,\\ \mathrm{div}B=0,\\ \mathrm{div}D=q\end{array}\right. \tag{1.2.15}$$

where D is the electric displacement, E the electric field vector, H the magnetic field vector, B the magnetic induction, j the current density and q is the charge density; c is the velocity of light. They also imply the charge conservation law:

$$\partial_t q+\mathrm{div}j=0. \tag{1.2.16}$$

To close the system, one needs *constitutive equations* which link E, D, H, B and j.

Equations in vacuum

Consider here the case $j = 0$ and $q = 0$ (no current and no charge) and

$$D = \varepsilon E, \qquad B = \mu H, \tag{1.2.17}$$

where ε is the dielectric tensor and μ the tensor of magnetic permeability.

In vacuum, ε and μ are scalar and constant. After some normalization the equation reduces to

$$\begin{cases} \partial_t E - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \\ \mathrm{div} B = 0, \\ \mathrm{div} E = 0. \end{cases} \tag{1.2.18}$$

The first two equations imply that $\partial_t \mathrm{div} E = \partial_t \mathrm{div} B = 0$, therefore the constraints $\mathrm{div} E = \mathrm{div} B = 0$ are satisfied at all time if they are satisfied at time $t = 0$. This is why one can "forget" the divergence equation and focus on the evolution equations

$$\begin{cases} \partial_t E - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \end{cases} \tag{1.2.19}$$

Moreover, using that $\mathrm{curl}\,\mathrm{curl} = -\Delta \mathrm{Id} + \mathrm{grad}\,\mathrm{div}$, for divergence free fields the system is equivalent to the wave equation :

$$\partial_t^2 E - \Delta E = 0. \tag{1.2.20}$$

In 3×3 block form, the symbol of (1.2.19) is

$$i\tau \mathrm{Id} + i \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & -\xi_2 & \xi_3 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \tag{1.2.21}$$

The system is hyperbolic and for $\xi \neq 0$, the eigenvalues and eigenspaces are

$$\tau = 0, \qquad \mathbb{E}_0 = \{\xi \times \dot{E} = 0,\ \xi \times \dot{B} = 0\}, \tag{1.2.22}$$

$$\tau = \pm|\xi|, \qquad \mathbb{E}_\pm = \{\dot{E} \in \xi^\perp,\ \dot{B} = \mp \frac{\xi \times \dot{E}}{|\xi|}\}. \tag{1.2.23}$$

Crystal optics

With $j = 0$ and $q = 0$, we assume in (1.2.17) that μ is scalar but that ε is a positive definite symmetric matrix. In this case the system reads:

$$\text{(1.2.24)} \qquad \begin{cases} \partial_t(\varepsilon E) - \mathrm{curl} B = 0, \\ \partial_t B + \mathrm{curl} E = 0, \end{cases}$$

plus the constraint equations $\mathrm{div}(\varepsilon E) = \mathrm{div} B = 0$ which are again propagated from the initial conditions. We choose coordinate axes so that ε is diagonal:

$$\text{(1.2.25)} \qquad \varepsilon^{-1} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

with $\alpha_1 > \alpha_2 > \alpha_3$. Ignoring the divergence conditions, the characteristic equation and the polarization conditions are obtained as solutions of system

$$\text{(1.2.26)} \qquad L(\tau, \xi) \begin{pmatrix} \dot{E} \\ \dot{B} \end{pmatrix} := \begin{pmatrix} \tau \dot{E} & - & \varepsilon^{-1}(\xi \times \dot{B}) \\ \tau \dot{B} & + & \xi \times \dot{E} \end{pmatrix} = 0\,.$$

For $\xi \neq 0$, $\tau = 0$ is a double eigenvalue, with eigenspace $\mathbb{E}_0$ as in (1.2.22). Note that these modes are incompatible with the divergence conditions. The nonzero eigenvalues are given as solutions of

$$\dot{E} = \varepsilon^{-1}(\frac{\xi}{\tau} \times \dot{B})\,, \quad (\tau^2 + \Omega(\xi)\varepsilon^{-1}\Omega(\xi))\dot{B} = 0$$

where $\Omega(\xi)$ is given in (1.2.21). Introduce

$$A(\xi) := \Omega(\xi)\varepsilon^{-1}\Omega(\xi) = \begin{pmatrix} -\alpha_2\xi_3^2 & \alpha_3\xi_1\xi_2 & \alpha_2\xi_1\xi_3 \\ \alpha_3\xi_1\xi_2 & -\alpha_1\xi_3^2 - \alpha_3\xi_1^2 & \alpha_1\xi_2\xi_3 \\ \alpha_2\xi_1\xi_3 & \alpha_1\xi_2\xi_3 & -\alpha_1\xi_2^2 - \alpha_2\xi_1^2 \end{pmatrix}.$$

Then

$$\det(\tau^2 + A(\xi)) \;=\; \tau^2\big(\tau^4 - \Psi(\xi)\tau^2 + |\xi|^2\,\Phi(\xi)\big)$$

with

$$\begin{cases} \Psi(\xi) = (\alpha_1 + \alpha_2)\xi_3^2 + (\alpha_2 + \alpha_3)\xi_1^2 + (\alpha_3 + \alpha_1)\xi_2^2 \\ \Phi(\xi) = \alpha_1\alpha_2\xi_3^2 + \alpha_2\alpha_3\xi_1^2 + \alpha_3\alpha_1\xi_2^2\,. \end{cases}$$

The nonvanishing eigenvalues are solutions of a second order equations in τ^2, of which the discriminant is

$$\Psi^2(\xi) - 4|\xi|^2\Phi(\xi) = P^2 + Q$$

with

$$P = (\alpha_1 - \alpha_2)\xi_3^2 + (\alpha_3 - \alpha_2)\xi_1^2 + (\alpha_3 - \alpha_1)\xi_2^2$$
$$Q = 4(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\xi_3^2\xi_2^2 \geq 0\,.$$

For a bi-axial crystal ε has three distinct eigenvalues and in general $P^2+Q \neq 0$. In this case, there are four simple eigenvalues

$$\pm\frac{1}{2}\Big(\Psi \pm \big(P^2 + Q\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}.$$

The corresponding eigenspace is made of vectors $(\dot{E}, \dot{B})$ such that $\dot{E} = \varepsilon^{-1}(\frac{\xi}{\tau} \times \dot{B})$ and $\dot{B}$ is an eigenvector of $A(\xi)$.

There are double roots exactly when $P^2 + Q = 0$, that is when

(1.2.27) $$\xi_2 = 0, \quad \alpha_1\xi_3^2 + \alpha_3\xi_1^2 = \alpha_2(\xi_1^2 + \xi_3^2) \,= \tau^2\,.$$

Laser - matter interaction

Still with $j = 0$ and $q = 0$ and B proportional to H, say $B = H$, the interaction light-matter is described through the relation

(1.2.28) $$D = E + P$$

where P is the polarization field. P can be given explicitly in terms of E, for instance in the *Kerr nonlinearity* model:

(1.2.29) $$P = |E|^2 E.$$

In other models P is given by an evolution equation:

(1.2.30) $$\frac{1}{\omega^2}\partial_t^2 P + P - \alpha|P|^2 P = \gamma E$$

harmonic oscillators when $\alpha = 0$ or *anharmonic oscillators* when $\alpha \neq 0$.

In other models, P is given by *Bloch's equation* which come from a more precise description of the physical interaction of the light and the electrons at the quantum mechanics level.

With $Q = \partial_t P$, the equations (1.2.15) (1.2.30) can be written as a first order 12×12 system:

(1.2.31) $$\begin{cases} \partial_t E - \mathrm{curl} B + Q = 0, \\ \partial_t B + \mathrm{curl} E = 0, \\ \partial_t P - Q = 0, \\ \partial_t Q + \omega^2 P - \omega^2\gamma E - \omega^2\alpha|P|^2 P = 0. \end{cases}$$

The linearized system around $P = 0$ is the same equation with $\alpha = 0$. In this case the (full) symbol is the block matrix

$$i\tau\mathrm{Id} + \begin{pmatrix} 0 & i\Omega & 0 & \mathrm{Id} \\ -i\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \\ -\omega^2\gamma & 0 & \omega^2 & 0 \end{pmatrix}.$$

The characteristic equations read

$$\begin{cases} \tau(\dot{E} + \dot{P}) - \xi \times \dot{B} = 0, \\ \tau\dot{B} + \xi \times \dot{E} = 0, \\ (\omega^2 - \tau^2)\dot{P} = \omega^2\gamma\dot{E}, \qquad \dot{Q} = i\tau\dot{P}. \end{cases}.$$

The eigenvalue $\tau = 0$ has multiplicity 2 with eigenspace

$$\mathbb{E}_0 = \big\{\xi \times \dot{E} = 0,\ \xi \times \dot{B} = 0,\ \dot{P} = \gamma\dot{E},\ \dot{Q} = 0\big\}.$$

Next, one can remark that $\tau = \omega$ is not an eigenvalue. Thus, when $\tau \neq 0$, the characteristic system can be reduced to

$$\tau^2\big(1 + \frac{\gamma\omega^2}{\omega^2 - \tau^2}\big)\dot{E} + \xi \times (\xi \times \dot{E}) = 0$$

together with

$$\dot{B} = -\frac{\xi \times E}{\tau}, \quad \dot{P} = \frac{\gamma\omega^2}{\omega^2 - \tau^2}\dot{E}, \quad \dot{Q} = i\tau\dot{P}.$$

This means that $\tau^2\big(1 + \frac{\gamma\omega^2}{\omega^2-\tau^2}\big)$ is an eigenvalue of $\xi \times (\xi \times \cdot)$, thus the non vanishing eigenvalues are solutions of

$$\tau^2\big(1 + \frac{\gamma\omega^2}{\omega^2 - \tau^2}\big) = |\xi|^2. \tag{1.2.32}$$

Multiplying by $\omega^2 - \tau^2$, this yields a second order equation in τ^2. For $\xi \neq 0$, this yields four distinct real eigenvalues of multiplicity two, with eigenspace given by

$$\mathbb{E} = \Big\{\dot{E} \in \xi^\perp,\ \dot{B} = -\frac{\xi \times E}{\tau}, \quad \dot{P} = \frac{\gamma\omega^2}{\omega^2 - \tau^2}\dot{E}, \quad \dot{Q} = i\tau\dot{P}\Big\}.$$

Note that the lack of homogeneity of the system (1.2.31) (with $\alpha = 0$) is reflected is the lack of homogeneity of the dispersion relation (1.2.32). For wave or Maxwell's equations, the coefficient n^2 in the dispersion relation $n^2\tau^2 = |\xi|^2$ is called the index of the medium. For instance, in vacuum the index is $n_0 = 1$ with the choice of units made in (1.2.18). Indeed, $\frac{n}{n_0}$ is related to the propagation of light in the medium (whose proper definition is $\frac{d\tau}{d|\xi|}$). An interpretation of (1.2.32) is that the index and the speed of propagation depend on the frequency. In particular, this model can be used to describe the well known phenomenon of dispersion of light propagating in glass.

1.2.3 Magneto-hydrodynamics

A model

The equations of isentropic magnetohydrodynamics (MHD) appear in basic form as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u^t u) + \nabla p + H \times \operatorname{curl} H = 0 \\ \partial_t H + \operatorname{curl}(H \times u) = 0 \end{cases} \tag{1.2.33}$$

$$\operatorname{div} H = 0, \tag{1.2.34}$$

where $\rho \in \mathbb{R}$ represents density, $u \in \mathbb{R}^3$ fluid velocity, $p = p(\rho) \in \mathbb{R}$ pressure, and $H \in \mathbb{R}^3$ magnetic field. With $H \equiv 0$, (1.2.33) reduces to the equations of isentropic fluid dynamics.

Equations (1.2.33) may be put in conservative form using identity

$$H \times \operatorname{curl} H = (1/2)\operatorname{div}(|H|^2 I - 2H^t H)^{\mathrm{tr}} + H \operatorname{div} H \tag{1.2.35}$$

together with constraint (1.2.34) to express the second equation as

$$\partial_t(\rho u) + \operatorname{div}(\rho u^t u) + \nabla p + (1/2)\operatorname{div}(|H|^2 I - 2H^t H)^{\mathrm{tr}} = 0. \tag{1.2.36}$$

They may be put in symmetrizable (but no longer conservative) form by a further change, using identity

$$\operatorname{curl}(H \times u) = (\operatorname{div} u)H + (u \cdot \nabla)H - (\operatorname{div} H)u - (H \cdot \nabla)u \tag{1.2.37}$$

together with constraint (1.2.34) to express the third equation as

$$\partial_t H + (\operatorname{div} u)H + (u \cdot \nabla)H - (H \cdot \nabla)u = 0. \tag{1.2.38}$$

Multiple eigenvalues

The first order term of the linearized equations about (u, H) is

$$(1.2.39)\qquad \begin{cases} D_t\dot\rho + \rho \div \dot u \\ D_t\dot u + \rho^{-1}c^2\nabla\dot\rho + \rho^{-1}H \times \mathrm{curl}\dot H \\ D_t\dot H + (\div\dot u)H - H\cdot\nabla\dot u \end{cases}$$

with $D_t = \partial_t + u\cdot\nabla$ and $c^2 = dp/d\rho$. The associated symbol is

$$(1.2.40)\qquad \begin{cases} \tilde\tau\dot\rho + \rho(\xi\cdot\dot u) \\ \tilde\tau\dot u + \rho^{-1}c^2\dot\rho\xi + \rho^{-1}H\times(\xi\times\dot H) \\ \tilde\tau\dot H + (\xi\cdot\dot u)H - (H\cdot\xi)\dot u \end{cases}$$

with $\tilde\tau = \tau + u\cdot\xi$. We use here the notation $\xi = (\xi_1, \xi_2, \xi_3)$ for the spatial frequencies and

$$\xi = |\xi|\,\hat\xi\,, \quad u_\| = \hat\xi\cdot u\,, \quad u_\perp = u - u_\|\hat\xi = -\hat\xi\times(\hat\xi\times u)\,.$$

We write (1.2.40) in the general form $\tau\mathrm{Id} + A(U,\xi)$ with parameters $U = (\rho, u, H)$. The eigenvalue equation $A(U,\xi)\dot U = \lambda\dot U$ reads

$$(1.2.41)\qquad \begin{cases} \tilde\lambda\dot\rho = \rho\dot u_\|, \\ \rho\tilde\lambda\dot u_\| = c^2\dot\rho + H_\perp\cdot\dot H_\perp, \\ \rho\tilde\lambda\dot u_\perp = -H_\|\dot H_\perp, \\ \tilde\lambda\dot H_\perp = \dot u_\| H_\perp - H_\|\dot u_\perp, \\ \tilde\lambda\dot H_\| = 0, \end{cases}$$

with $\tilde\lambda = \lambda - (u\dot\xi)$. The last condition decouples. On the space

$$(1.2.42)\qquad \mathbb{E}_0(\xi) = \big\{\dot\rho = 0,\ \dot u = 0,\ \dot H_\perp = 0\big\},$$

A is equal to $\lambda_0 := u\cdot\xi$. From now on we work on $\mathbb{E}_0^\perp = \{\dot H_\| = 0\}$ which is invariant by $A(U,\xi)$.

Consider $v = H/\sqrt\rho$, $\dot v = \dot H/\sqrt\rho$ and $\dot\sigma = \dot\rho/\rho$. The characteristic system reads:

$$(1.2.43)\qquad \begin{cases} \tilde\lambda\dot\sigma = \dot u_\| \\ \tilde\lambda\dot u_\| = c^2\dot\sigma + v_\perp\cdot\dot v_\perp \\ \tilde\lambda\dot u_\perp = -v_\|\dot v_\perp \\ \tilde\lambda\dot v_\perp = \dot u_\| v_\perp - v_\|\dot u_\perp \end{cases}$$

Take a basis of $\xi^\perp$ such that $v_\perp = (b, 0)$ and let $a = v_\parallel$. In such a basis, the matrix of the system reads

$$\tilde\lambda - \tilde{\mathcal{A}} := \begin{bmatrix} \tilde\lambda & -1 & 0 & 0 & 0 & 0 \\ -c^2 & \tilde\lambda & 0 & 0 & -b & 0 \\ 0 & 0 & \tilde\lambda & 0 & a & 0 \\ 0 & 0 & 0 & \tilde\lambda & 0 & a \\ 0 & -b & a & 0 & \tilde\lambda & 0 \\ 0 & 0 & 0 & a & 0 & \tilde\lambda \end{bmatrix} \tag{1.2.44}$$

The characteristic roots satisfy

$$(\tilde\lambda^2 - a^2)\big((\tilde\lambda^2 - a^2)(\tilde\lambda^2 - c^2) - \tilde\lambda^2 b^2\big) = 0\,. \tag{1.2.45}$$

Thus, either

$$\tilde\lambda^2 = a^2 \tag{1.2.46}$$

$$\tilde\lambda^2 = c_f^2 := \frac{1}{2}\Big(c^2 + h^2) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\Big) \tag{1.2.47}$$

$$\tilde\lambda^2 = c_s^2 := \frac{1}{2}\Big(c^2 + h^2) - \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\Big) \tag{1.2.48}$$

with $h^2 = a^2 + b^2 = |H|^2/\rho$.

With $P(X) = (X - a^2)(X - c^2) - b^2X$, $\{P \le 0\} = [c_s^2, c_f^2]$ and $P(X) \le 0$ for $X \in [\min(a^2, c^2), \max(a^2, c^2)]$. Thus,

$$c_f^2 \ge \max(a^2, c^2) \ge a^2 \tag{1.2.49}$$

$$c_s^2 \le \min(a^2, c^2) \le a^2 \tag{1.2.50}$$

1. The case $v_\perp \ne 0$ i.e. $w = \hat\xi \times v \ne 0$. Thus, the basis such that (1.2.44) holds is smooth in ξ. In this basis, $w = (0, b)$, $b = |v_\perp| > 0$.

1.1 The spaces

$$\mathbb{E}_\pm(\hat\xi) = \{\dot\sigma = 0, \dot u_\parallel = 0,\ \dot v_\perp \in \mathbb{C}(\hat\xi \times v),\ \dot u_\perp = \mp \dot v_\perp\}$$

are invariant for $\tilde{\mathcal{A}}$ and

$$\tilde{\mathcal{A}} = \pm a \quad \text{on} \quad \mathbb{E}_\pm\,. \tag{1.2.51}$$

1.2 In $(\mathbb{E}_+ \oplus \mathbb{E}_-)^\perp$, which is invariant, the matrix of $\tilde{\mathcal{A}}$ is

$$\tilde{\mathcal{A}}_0 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ c^2 & 0 & 0 & -b \\ 0 & 0 & 0 & -a \\ 0 & -b & a & 0 \end{bmatrix} \tag{1.2.52}$$

Since $P(c^2) = -b^2c^2 < 0$, there holds $c_s^2 < c^2 < c_f^2$.

1.2.1) Suppose that $a \neq 0$. Then, $P(a^2) = -a^2c^2 < 0$ and $c_s^2 < a^2 < c_f^2$. Thus, all the eigenvalues are simple. Moreover, $c_s^2 c_f^2 = a^2c^2$ and $c_s^2 > 0$. The space

$$\mathbb{F}_{\tilde{\lambda}} = \Big\{ \tilde{\lambda}\dot{\sigma} = \dot{u}_{\|},\ \dot{u}_{\|} = \frac{\tilde{\lambda} v_\perp \cdot \dot{v}_\perp}{\tilde{\lambda}^2 - c^2},\ \dot{u}_\perp = \frac{-a\dot{v}_\perp}{\tilde{\lambda}},\ \dot{v}_\perp =\in \mathbb{C}v_\perp \Big\}$$

is an eigenspace associated to the eigenvalue $\tilde{\lambda}$ when $\tilde{\lambda} = \pm c_f$ and $\tilde{\lambda} = \pm c_s$. Here $\dot{u}_1$ and $\dot{v}_1$ denote the first component of $\dot{u}_\perp$ and $\dot{v}_\perp$ respectively in the basis $(v_\perp, w)$.

1.2.1) Suppose that a is close to 0. Since $c_f^2 > c^2 > 0$, the spaces $\mathbb{F}_{\pm c_f}$ are still eigenspaces associated to the eigenvalues $\tilde{\lambda} = \pm c_f$.

By direct computations:

$$c_s^2 = \frac{c^2a^2}{c^2 + h^2} + O(a^4)\,.$$

Therefore,

$$\frac{c_s^2}{a^2} \to \frac{c^2}{c^2 + h^2} > 0 \quad \text{as } a \to 0\,.,.$$

Therefore, $\tilde{c}_s = \frac{a}{|a|}c_s$ is an analytic function of a (and $b \neq 0$) near $a = 0$ and

$$\widetilde{\mathbb{F}}_{\pm,s}(a,b) = \mathbb{F}_{\pm\tilde{c}_s}$$

are analytic determinations of Eigenspaces, associated to the eigenvalues $\pm\tilde{c}_s$. Moreover, the values at $a = 0$ are

$$\widetilde{\mathbb{F}}_{\pm,s}(0,b) = \big\{ \dot{\sigma} = \frac{-v_\perp \cdot \dot{v}_\perp}{c^2},\ \dot{u}_{\|} = 0,\ \dot{u}_\perp = \frac{\mp\sqrt{c^2 + b^2}}{c}\dot{v}_\perp \in \mathbb{C}v_\perp \big\}$$

and $\widetilde{\mathbb{F}}_{+,s} \cap \widetilde{\mathbb{F}}_{-,s} = \{0\}$, thus we still have an analytic diagonalization of $\tilde{\mathcal{A}}_0$.

2. Suppose now that b is close to zero. At $b = 0$, the eigenvalues of $\tilde{\mathcal{A}}$ are $\pm c$ (simple) and $\pm h$ (double). **Assume that** $c^2 \neq h^2$. Note that when $b = 0$, then $|a| = h$ and

$$\begin{array}{ll} \text{when } c^2 > h^2 \ : & c_f = c, \quad c_s = h, \\ \text{when } c^2 < h^2 \ : & c_f = h, \quad c_s = c. \end{array}$$

2.1 The eigenvalues close to $\pm c$ remain simple.

2.2 We look for the eigenvalues close to h. The characteristic equation implies that

$$\text{(1.2.53)} \qquad \begin{cases} c^2 \dot\sigma = \tilde\lambda \dot u_{\|} - v_\perp \cdot \dot v_\perp \\ (\tilde\lambda^2 - c^2)\dot u_{\|} = \tilde\lambda v_\perp \cdot \dot v_\perp \end{cases}$$

Eliminating $\dot u_{\|}$, we are left with the 4×4 system in $\xi^\perp \times \xi^\perp$:

$$\text{(1.2.54)} \qquad \begin{cases} \tilde\lambda \dot u_\perp = -a \dot v_\perp \\ \tilde\lambda \dot v_\perp = -a \dot u_\perp + \dfrac{\tilde\lambda}{\tilde\lambda^2 - c^2}(v_\perp \otimes v_\perp)\dot v_\perp . \end{cases}$$

Thus,

$$(\tilde\lambda^2 - a^2)\dot v_\perp = \frac{\tilde\lambda^2}{\tilde\lambda^2 - c^2}(v_\perp \otimes v_\perp)\dot v_\perp .$$

Recall that $|v_\perp| = b$ is small. We recover 4 *smooth eigenvalues*

$$\text{(1.2.55)} \qquad \pm a \, , \qquad \pm\sqrt{a^2 + O(b^2)} = \pm(a + O(b^2)) \, .$$

(remember that $a = \pm h + O(b^2)$. However, the eigenspaces are not smooth in v, since they are $\mathbb{R} v_\perp$ and $\mathbb{R}\hat\xi \times v_\perp$ and have no limit at $v_\perp \to 0$.

Summing up, we have proved the following.

Lemma 1.2.1. *Assume that* $c^2 = dp/d\rho > 0$. *The eigenvalues of* $A(U, \xi)$ *are*

$$\text{(1.2.56)} \qquad \begin{cases} \lambda_0 = \xi \cdot u \\ \lambda_{\pm 1} = \lambda_0 \pm c_s(\hat\xi)|\xi| \\ \lambda_{\pm 2} = \lambda_0 \pm (\xi \cdot H)/\sqrt{\rho} \\ \lambda_{\pm 3} = \lambda_0 \pm c_f(\hat\xi)|\xi| \end{cases}$$

with $\hat{\xi} = \xi/|\xi|$ *and*

$$c_f^2(\hat{\xi}) := \frac{1}{2}\Big(c^2 + h^2) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\Big) \tag{1.2.57}$$

$$c_s^2(\hat{\xi}) := \frac{1}{2}\Big(c^2 + h^2) - \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\Big) \tag{1.2.58}$$

where $h^2 = |H|^2/\rho$, $b^2 = |\hat{\xi} \times H|^2/\rho$.

Lemma 1.2.2. *Assume that* $0 < |H|^2 \neq \rho c^2$ *where* $c^2 = dp/d\rho > 0$.

i) When $\xi \cdot H \neq 0$ *and* $\xi \times H \neq 0$, *the eigenvalues of* $A(U, \xi)$ *are simple.*

ii) Near the manifold $\xi \cdot H = 0$, $\xi \neq 0$, *the eigenvalues* $\lambda_{\pm 3}$ *are simple. The other eigenvalues can be labeled so that they are smooth and coincide exactly on* $\{\xi \cdot H = 0\}$. *Moreover, there is a smooth basis of eigenvectors.*

iii) Near the manifold $\xi \times H = 0$, $\xi \neq 0$, λ_0 *is simple. When* $|H|^2 < \rho c^2$ *[resp.* $|H|^2 > \rho c^2$ *],* $\lambda_{\pm 3}$ *[resp.* $\lambda_{\pm 1}$*] are simple;* $\lambda_{+2} \neq \lambda_{-2}$ *are double, equal to* $\lambda_{\pm 1}$ *[resp.* $\lambda_{\pm 3}$ *] depending on the sign of* $\xi \cdot H$. *They are smooth, but there is no smooth basis of eigenvectors.*

1.2.4 Elasticity

The linear wave equation in an elastic homogeneous medium is a second order constant coefficients 3×3 system

$$\partial_t^2 v - \sum_{j,k=1}^{3} A_{j,k} \partial_{x_j} \partial_{x_k} v = f \tag{1.2.59}$$

where the $A_{j,k}$ are 3×3 real matrices. In anisotropic media, the form of the matrices $A_{j,k}$ is complicated (it may depend upon 21 parameters). The basic hyperbolicity condition is that

$$A(\xi) := \sum \xi_j \xi_k A_{j,k} \tag{1.2.60}$$

is symmetric and positive definite for $\xi \neq 0$.

In the isotropic case

$$\sum_{j,k=1}^{3} A_{j,k} \partial_{x_j} \partial_{x_k} v = 2\lambda \Delta_x v + \mu \nabla_x (\mathrm{div}_x v). \tag{1.2.61}$$

The hyperbolicity condition is that $\lambda > 0$ and $2\lambda + \mu > 0$.

Chapter 2

Constant Coefficient Systems. Fourier Synthesis

In this chapter, we review the resolution of constant coefficient equations by Fourier synthesis. Our first objective is to give an obvious sufficient condition (Assumption 2.1.8) for the well posed-ness of the Cauchy problem in L^2 or H^s (Theorem 2.1.9). The second important content of the chapter is the introduction of the notion of hyperbolicity, from an analysis of the general condition. Next, we briefly discuss. different notions of hyperbolicity, but we confine ourselves to the elementary cases.

2.1 The method

In this chapter, we consider equations (or systems)

$$\begin{cases} \partial_t u + A(\partial_x) u = f & \text{on } [0, T] \times \mathbb{R}^d, \\ u_{|t=0} = h & \text{on } \mathbb{R}^d. \end{cases} \tag{2.1.1}$$

where A is a differential operator (or system) with *constant coefficients* :

$$A(\partial_x) = \sum A_\alpha \partial_x^\alpha u \tag{2.1.2}$$

2.1.1 The Fourier transform

Notations 2.1.1. The (spatial) Fourier transform $\mathscr{F}$ is defined for $u \in L^1(\mathbb{R}^d)$ by

$$\mathscr{F} u(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x) dx. \tag{2.1.3}$$

We also use the notations $\hat{u}(\xi)$ for the Fourier transform $\mathscr{F}u(\xi)$. If $\hat{u} \in L^1(\mathbb{R}^d)$ the inverse transformation $\mathscr{F}^{-1}$ is:

$$\mathscr{F}^{-1}\hat{u}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{u}(\xi)d\xi. \tag{2.1.4}$$

We denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz class, by $\mathscr{S}'(\mathbb{R}^d)$ the space of temperate distributions and by $\mathscr{E}'(\mathbb{R}^d)$ the space of distributions with compact support.

Theorem 2.1.2 (Reminders)**.**

i) $\mathscr{F}$ *is a one to one mapping from the Schwartz class* $\mathscr{S}(\mathbb{R}^d)$ *onto itself with reciprocal* $\mathscr{F}^{-1}$.

ii) $\mathscr{F}$ *and* $\mathscr{F}^{-1}$ *extend as bijections from the space of temperate distributions* $\mathscr{S}'(\mathbb{R}^d)$ *onto itself. Moreover, for* $u \in \mathscr{S}'$ *and* $v \in \mathscr{S}$ *there holds*

$$\big\langle \hat{u}, v\big\rangle_{\mathscr{S}'\times\mathscr{S}} \ = \ \big\langle u, \hat{v}\big\rangle_{\mathscr{S}'\times\mathscr{S}} \tag{2.1.5}$$

iii) Plancherel's theorem : $\mathscr{F}$ *is an isomorphism from* L^2 *onto itself and*

$$\int u(x)\overline{v}(x)dx = \frac{1}{(2\pi)^d}\int \hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi. \tag{2.1.6}$$

In particular

$$\big\|\hat{u}\big\|_{L^2} = \sqrt{(2\pi)^n}\ \big\|u\big\|_{L^2}. \tag{2.1.7}$$

iv) For $u \in \mathscr{S}'(\mathbb{R}^d)$ *there holds :*

$$\widehat{\partial_{x_j}u}(\xi) = i\xi_j\hat{u}(\xi) \tag{2.1.8}$$

$$\widehat{x_ju}(\xi) = -i\partial_{\xi_j}\hat{u}(\xi) \tag{2.1.9}$$

v) For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ *is the space of temperate distributions* u *such that* $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^d)$. *It is an Hilbert space equipped with the norm*

$$\big\|u\big\|^2_{H^s} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2d\xi. \tag{2.1.10}$$

Combining *ii*) and *iii*) implies that for $u \in \mathscr{S}'$ and $v \in \mathscr{S}$ there holds

$$\big\langle u, \overline{v}\big\rangle_{\mathscr{S}'\times\mathscr{S}} \ = \ \frac{1}{(2\pi)^d}\big\langle \hat{u}, \overline{\hat{v}}\big\rangle_{\mathscr{S}'\times\mathscr{S}} \tag{2.1.11}$$

The *spectrum* of u is the support of $\hat{u}$.

When u also depends on time, we let $\mathscr{F}$ act for all fixed t and use the following notations:

Notations 2.1.3. If u is a continuous (or measurable) function of time with values in a space of temperate distributions, $\hat{u}$ or $\mathscr{F}u$ denotes the function defined for all (or almost all) t by

$$\hat{u}(t, \xi) = \mathscr{F}(u(t, \cdot))(\xi).$$

In particular, the identity

$$\big\langle \hat{u}, v \big\rangle_{\mathscr{S}' \times \mathscr{S}} \;=\; \big\langle u, \hat{v} \big\rangle_{\mathscr{S}' \times \mathscr{S}} \tag{2.1.12}$$

is satisfied for $u \in \mathscr{S}'(\mathbb{R}^{1+d})$ and $v \in \mathscr{S}(\mathbb{R}^{1+d})$.

2.1.2 Solving the evolution equation (2.1.1)

Lemma 2.1.4. *If $u \in \mathscr{S}'(\mathbb{R}^d)$ then*

$$\widehat{A(\partial_x)u}(\xi) = A(i\xi)\hat{u}(\xi), \qquad A(\xi) = \sum A_\alpha (i\xi)^\alpha. \tag{2.1.13}$$

Remark 2.1.5. In the scalar case, this means that $\mathscr{F}$ diagonalizes $A(\partial_x)$, with eigenfunctions $x \mapsto e^{i\xi \cdot x}$ and eigenvalues $A(i\xi)$:

$$A(\partial_x) e^{i\xi \cdot x} = A(i\xi) e^{i\xi \cdot x}.$$

For systems, there is a similar interpretation.

Using (2.1.12) immediately implies the following:

Lemma 2.1.6. *If $u \in L^1(]0, T[; H^s(\mathbb{R}^d))$, then in the sense of distributions,*

$$\widehat{\partial_t u}(t, \xi) = \partial_t \hat{u}(t, \xi). \tag{2.1.14}$$

Corollary 2.1.7. *For $u \in C^0([0, T]; H^s(\mathbb{R}^d))$ and $f \in L^1([0, T]; H^{s'}(\mathbb{R}^d))$, the equation (2.1.1) is equivalent to*

$$\begin{cases} \partial_t \hat{u} + A(i\xi)\hat{u} = \hat{f} & \text{on } [0, T] \times \mathbb{R}^d, \\ \hat{u}_{|t=0} = \hat{h} & \text{on } \mathbb{R}^d. \end{cases} \tag{2.1.15}$$

The solution of (2.1.15) is

$$\hat{u}(t, \xi) = e^{-tA(i\xi)}\hat{h}(\xi) + \int_0^t e^{(t'-t)A(i\xi)} \hat{f}(t', \xi) dt'. \tag{2.1.16}$$

Question : show that the right hand side of (2.1.16) defines a temperate distribution in ξ. If this is correct, then the inverse Fourier transform defines a function u with values in $\mathscr{S}'$, which by construction is a solution of (2.1.1).

This property depends on the behavior of the exponentials $e^{-tA(i\xi)}$ when $|\xi| \to \infty$. The simplest case is the following:

Assumption 2.1.8. *There is a function $C(t)$ bounded on all interval $[0,T]$, such that*

$$\forall t \geq 0,\ \forall \xi \in \mathbb{R}^d \qquad \big|e^{-tA(i\xi)}\big| \leq C(t). \tag{2.1.17}$$

Theorem 2.1.9. *Under the Assumption 2.1.8, for $h \in H^s(\mathbb{R}^d)$ and $f \in L^1([0,T]; H^s(\mathbb{R}^d))$, the formula (2.1.16) defines a fonction $u \in C^0([0,T]; H^s(\mathbb{R}^d)$ which satisfies (2.1.1) together with the bounds*

$$\big\|u(t)\big\|_{H^s} \leq C(t)\big\|h\big\|_{H^s} + \int_0^t C(t-t')\big\|f(t')\big\|_{H^s} dt'. \tag{2.1.18}$$

Proof. Assumption (2.1.18) implies that

$$\big|e^{-tA(i\xi)}\hat{h}(\xi)\big| \leq C(t)\big|\hat{h}(\xi)\big|.$$

Thus, by Lebesgues' dominated convergence theorem, if $h \in L^2$, the mapping $t \mapsto \hat{u}_0(t,\cdot) = \hat{u}_0(t,\cdot) = e^{-tA(i\cdot)}\hat{h}(\cdot)$ is continuous from $[0,+\infty[$ to $L^2(\mathbb{R}^d)$. Thus, $u_0 = \mathscr{F}^{-1}u \in C^0([0,+\infty[; L^2(\mathbb{R}^d)$. Moreover:

$$\big\|u(t)\big\|_{L^2} = \frac{1}{\sqrt{(2\pi)^n}}\big\|\hat{u}(t)\big\|_{L^2} \leq \frac{C(t)}{\sqrt{(2\pi)^n}}\big\|\hat{h}\big\|_{L^2} = C(t)\big\|h\big\|_{L^2}.$$

Similarly, the function $\hat{v}(t,t',\xi) = e^{(t'-t)A(i\xi)}\hat{f}(t',\xi)$ satisfies

$$\big\|\hat{v}(t,t',\cdot)\big\|_{L^2} \leq C(t-f')\big\|\hat{f}(t',\cdot)\big\|_{L^2}.$$

Therefore, Lebesgues' dominated convergence theorem implies that

$$\hat{u}_1(t,\xi) = \int_0^t \hat{v}(t,t',\xi)dt'$$

belongs to $C^0([0,T]; L^2(\mathbb{R}^d))$ and satisfies

$$\big\|\hat{u}_1(t)\big\|_{L^2} \leq \int_0^t C(t-t')\big\|\hat{f}(t')\big\|_{L^2} dt'.$$

Taking the inverse Fourier transform, $u_1 = \mathscr{F}^{-1}\hat{u}_1$ belongs to $C^0([0,T]; L^2(\mathbb{R}^d))$ and satisfies

$$\big\|u_1(t)\big\|_{L^2} \leq \int_0^t C(t-t')\big\|f(t')\big\|_{L^2} dt'.$$

There are completely similar estimates in H^s. Adding u_0 and u_1, the theorem follows. $\square$

2.2 Examples

2.2.1 The heat equation

It reads

$$\partial_t u - \Delta_x u = f, \quad u_{|t=0} = h. \tag{2.2.1}$$

On the Fourier side, it is equivalent to

$$\partial_t \hat{u} + |\xi|^2 \hat{u} = \hat{f}, \quad \hat{u}_{|t=0} = \hat{h}. \tag{2.2.2}$$

and the solution is

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{h}(\xi) + \int_0^t e^{(t'-t)|\xi|^2}\hat{f}(t',\xi)dt'. \tag{2.2.3}$$

Remark 2.2.1. The Theorem 2.1.9 can be applied, showing that the Cauchy problem is well posed. However, it does not give the optimal results : the smoothing properties of the heat equation can be also deduced from the explicit formula (2.2.3), using the exponential decay of $e^{-t|\xi|^2}$ as $|\xi| \to \infty$, while Theorem 2.1.9 only uses that it is uniformly bounded.

2.2.2 Schrödinger equation

A basic equation from quantum mechanics is:

$$\partial_t u - i\Delta_x u = f, \quad u_{|t=0} = h. \tag{2.2.4}$$

Note that this equation is also very common in optics and in many other fields, as it appears as a canonical model in the so-called *paraxial approximation*, used for instance to model the dispersion of light along long propagations.

The Fourier transform of (2.2.4) reads

$$\partial_t \hat{u} - i|\xi|^2 \hat{u} = \hat{f}, \quad \hat{u}_{|t=0} = \hat{h}. \tag{2.2.5}$$

The solution is

$$\hat{u}(t,\xi) = e^{it|\xi|^2}\hat{h}(\xi) + \int_0^t e^{(t-t')|\xi|^2}\hat{f}(t',\xi)dt'. \tag{2.2.6}$$

Since $\left|e^{it|\xi|^2}\right| = 1$, the Theorem 2.1.9 can be applied, both for $t \geq 0$ and $t \leq 0$, showing that the Cauchy problem is well posed in Sobolev spaces.

2.2.3 The wave equation

It is second order, but the idea is similar.

$$\partial_t^2 u - \Delta_x u = f, \quad u_{|t=0} = h_0, \quad \partial_t u_{|t=0} = h_1. \tag{2.2.7}$$

By Fourier

$$\partial_t^2 \hat{u} + |\xi|^2 \hat{u} = \hat{f}, \quad \hat{u}_{|t=0} = \hat{h}_0, \quad \partial_t \hat{u}_{|t=0} = \hat{h}_1. \tag{2.2.8}$$

$$\begin{aligned} \hat{u}(t,\xi) = \cos(t|\xi|)\hat{h}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{h}_1(\xi) \\ + \int_0^t \frac{\sin((t-t')|\xi|)}{|\xi|}\hat{f}(t',\xi)dt'. \end{aligned} \tag{2.2.9}$$

Theorem 2.2.2. *For $h_0 \in H^{s+1}(\mathbb{R}^d)$, $h_1 \in H^s(\mathbb{R}^d)$ and $f \in L^1([0,T]; H^s(\mathbb{R}^d))$, (2.1.16) defines $u \in C^0([0,T]; H^{s+1}(\mathbb{R}^d)$ such that $\partial_t u \in C^0([0,T]; H^{s+1}(\mathbb{R}^d)$, u is a solution of (2.2.7) and*

$$\begin{aligned} \big\|u(t)\big\|_{H^{s+1}} \le \big\|h_0\big\|_{H^{s+1}} + 2(1+t)\big\|h_1\big\|_{H^s} \\ + 2(1+t)\int_0^t \big\|f(t')\big\|_{H^s} dt'. \end{aligned} \tag{2.2.10}$$

$$\big\|\partial_t u(t), \partial_{x_j} u(t)\big\|_{H^s} \le \big\|h_0\big\|_{H^{s+1}} + \big\|h_1\big\|_{H^{s+1}} + \int_0^t \big\|f(t')\big\|_{H^s} dt'. \tag{2.2.11}$$

Preuve. The estimates (2.2.10) follow from the inequalities

$$\big|\cos(t|\xi|)\big| \le 1, \qquad \Big|\frac{\sin(t|\xi|)}{|\xi|}\Big| \le \min\{t, \frac{1}{|\xi|}\} \le \frac{\sqrt{2}(1+t)}{\sqrt{1+|\xi|^2}}. \tag{2.2.12}$$

Moreover,

$$\begin{aligned} \partial_t \hat{u}(t,\xi) = -|\xi|\sin(t|\xi|)\hat{h}_0(\xi) + \cos(t|\xi|)\hat{h}_1(\xi) \\ + \int_0^t \cos((t-t')|\xi|)\hat{f}(t',\xi)dt'. \end{aligned} \tag{2.2.13}$$

Bounding $|\sin|$ and $|\cos|$ by 1 implies the estimates (2.2.11) for $\partial_t u$. Similarly, the Fourier transform of $v_j = \partial_{x_j} u$ is

$$\begin{aligned} \hat{v}_j(t,\xi) = i\xi_j\cos(t|\xi|)\hat{h}_0(\xi) + i\frac{\xi_j\sin(t|\xi|)}{|\xi|}\hat{h}_1(\xi) \\ + i\int_0^t \frac{\xi_j\sin((t-t')|\xi|)}{|\xi|}\hat{f}(t',\xi)dt'. \end{aligned} \tag{2.2.14}$$

Since $|\sin|$, $|\cos|$ and $\frac{|\xi_j|}{|\xi|}$ are bounded by 1, this implies that $\partial_{x_j} u$ satisfies (2.2.11). $\square$

2.3 First order systems: hyperbolicity

2.3.1 The general formalism

Consider a $N \times N$ systems

$$\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u = f, \qquad u_{|t=0} = h, \tag{2.3.1}$$

where $u(t,x)$, $f(t,x)$ et $h(x)$ take their values in $\mathbb{R}^N$ (or $\mathbb{C}^N$). The coordinates are denoted by $(u_1, \ldots, u_N)$, $(f_1, \ldots, f_N)$, $(h_1, \ldots, h_N)$. The A_j are $N \times N$ constant matrices.

After Fourier transform the system reads:

$$\partial_t \hat{u} + iA(\xi)\hat{u} = \hat{f}, \qquad \hat{u}_{|t=0} = \hat{h}, \tag{2.3.2}$$

with

$$A(\xi) = \sum_{j=1}^{n} \xi_j A_j. \tag{2.3.3}$$

The solution of (2.3.2) is given by

$$\hat{u}(t,\xi) = e^{-itA(\xi)}\hat{h}(\xi) + \int_0^t e^{i(t'-t)A(\xi)} \hat{f}(t',\xi) dt'. \tag{2.3.4}$$

2.3.2 Strongly hyperbolic systems

Following the general discussion, the problem is to give estimates for the exponentials $e^{-itA(\xi)} = e^{iA(-t\xi)}$. The next lemma is immediate.

Lemma 2.3.1. *For the exponential $e^{iA(\xi)}$ to have at most a polynomial growth when $|\xi| \to \infty$, it is necessary that for all $\xi \in \mathbb{R}^d$ the eigenvalues of $A(\xi)$ are real.*

In this case, the system is said to be hyperbolic.

From Theorem 2.1.9 we know that the problem is easily solved when the condition (2.1.17) is satisfied. Taking into account the homogeneity of $A(\xi)$, leads to the following definition.

Definition 2.3.2. *The system* (2.3.1) *is said to be strongly hyperbolic if there is a constant C such that*

$$\forall \xi \in \mathbb{R}^d, \qquad \big|e^{iA(\xi)}\big| \leq C. \tag{2.3.5}$$

The norm used on the space of $N \times N$ matrices is irrelevant. To fix ideas, on can equip $\mathbb{C}^N$ with the usual norm

$$\big|u\big| = \Big(\sum_{k=1}^{N} |u_k|^2\Big)^{\frac{1}{2}}. \tag{2.3.6}$$

The associated norm for $N \times N$ matrices M is

$$\big|M\big| = \sup_{\{u \in \mathbb{C}^N, |u|=1\}} \big|Mu\big| \tag{2.3.7}$$

By homogeneity, (2.3.5) is equivalent to

$$\forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^d, \qquad \big|e^{itA(\xi)}\big| \leq C. \tag{2.3.8}$$

Lemma 2.3.3. *The system is strongly hyperbolic if and only if*
i) for all $\xi \in \mathbb{R}^d$ the eigenvalues of $A(\xi)$ are real and semi-simple,
ii) there is a constant C such that for all $\xi \in \mathbb{R}^d$ the eigenprojectors of $A(\xi)$ have a norm bounded by C.

Proof. Suppose that the system is strongly hyperbolic. If $A(\xi)$ has a non real or real and not semi-simple eivengalue then $e^{iA(\pm t\xi)}$ is not bounded as $t \to \infty$. Thus A satisfies i). Moreover, the eigenprojector associated to the eigenvalue λ is

$$\Pi = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-is\lambda}\, e^{iA(s\xi)}\, ds\,.$$

Thus (2.3.8) implies that $|\Pi| \leq C$.

Conversely, if A satisfies i) then

$$A(\xi) = \sum_j \lambda_j(\xi) \Pi_j(\xi) \tag{2.3.9}$$

where the λ_j's are the real eigenvalues with eigenprojectors $\Pi_j(\xi)$. Thus

$$e^{iA(\xi)} = \sum_j e^{i\lambda_j(\xi)} \Pi_j(\xi). \tag{2.3.10}$$

Therefore, ii) implies that $|e^{iA(\xi)}| \leq NC$. □

2.3.3 Symmetric hyperbolic systems

A particular case of matrices with real eigenvalues and bounded exponentials are real symmetric (or complex self-adjoint). More generally, it is sufficient that they are self- adjoint for *some* hermitian scalar product on $\mathbb{C}^N$.

Definition 2.3.4. *i) The system* (2.3.1) *is said to be hyperbolic symmetric if for all j the matrices A_j are self adjoint.*

ii) The system (2.3.1) *is said to be hyperbolic symmetrizable if there exists a self-adjoint matrix S, positive definite, such that for all j the matrices SA_j are self adjoint.*

In this case S is called a symmetrizer.

Theorem 2.3.5. *If the system is hyperbolic symmetrizable it is strongly hyperbolic.*

Proof. **a)** If S is self-adjoint, there is a unitary matrix Ω such that

$$S = \Omega^{-1} D \Omega, \qquad D = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \tag{2.3.11}$$

with $\lambda_k \in \mathbb{R}$. Therefore,

$$e^{itS} \Omega^{-1} e^{itD} \Omega, \qquad e^{itD} = \mathrm{diag}(e^{it\lambda_1}, \dots, e^{it\lambda_N}).$$

Because the λ_k are real, e^{itD} and hence e^{itS} are unitary. In particular, $\left| e^{itS} \right| = 1$.

b) If the system is symmetric, then for all $\xi \in \mathbb{R}^d$, $A(\xi)$ is self adjoint. Thus

$$\left| e^{iA(\xi)} \right| = 1. \tag{2.3.12}$$

c) Suppose that the system is symmetrizable, with symmetrizer S. Since S is definite positive, its eigenvalues are positive. Using (2.3.11), this allows to define

$$S^{\frac{1}{2}} = \Omega^{-1} D^{\frac{1}{2}} \Omega, \qquad D^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_N}). \tag{2.3.13}$$

There holds

$$A(\xi) = S^{-\frac{1}{2}} S^{-\frac{1}{2}} S A(\xi) S^{-\frac{1}{2}} S^{\frac{1}{2}} = S^{-\frac{1}{2}} B(\xi) S^{\frac{1}{2}}. \tag{2.3.14}$$

Thus $A(\xi)$ is conjugated to $B(\xi)$ and

$$e^{iA(\xi)} = S^{-\frac{1}{2}} e^{iB(\xi)} S^{\frac{1}{2}}. \tag{2.3.15}$$

Since $SA(\xi)$ is self-adjoint, $B(\xi) = S^{-\frac{1}{2}} SA(\xi) S^{-\frac{1}{2}}$ is also self-adjoint and $\left|e^{iB(\xi)}\right| = 1$. Therefore,

$$\left|e^{iA(\xi)}\right| \leq \left|S^{-\frac{1}{2}}\right| \left|S^{\frac{1}{2}}\right| \tag{2.3.16}$$

implying that (2.3.5) is satisfied. □

Example 2.3.6. Maxwell equations, linearized Euler equations, equations of acoustics, linearized MHD introduced in Chapter 1 are hyperbolic symmetric or symmetrizable.

2.3.4 Smoothly diagonalizable systems, hyperbolic systems with constant multiplicities

Property *i)* in Lemma 2.3.3 says that for all ξ, $A(\xi)$ has only real eigenvalues and can be diagonalized. This does not necessarily imply strong hyperbolicity: the existence of a uniform bound for the eigenprojectors for $|\xi| = 1$ is a genuine additional condition. For extensions to systems with variable coefficients, an even stronger condition is required :

Definition 2.3.7. *The system* (2.3.1) *is said to be smoothly diagonalizable is there are real valued* $\lambda_j(\xi)$ *and projectors* $\Pi_j(\xi)$ *which are real analytic functions of* ξ *on the unit sphere, such that* $A(\xi) = \sum \lambda_j(\xi)\Pi_j(\xi)$.

In this case, continuity of the Π_j implies boundedness on S^{d-1} and therefore:

Lemma 2.3.8. *If* (2.3.1) *is smoothly diagonalizable, then it is strongly hyperbolic.*

Definition 2.3.9. *The system* (2.3.1) *is said to be strictly hyperbolic if for all* $\xi \neq 0$, $A(\xi)$ *has* N *distinct real eigenvalues.*

It is said to be hyperbolic with constant multiplicities if for all $\xi \neq 0$, $A(\xi)$ *has only real semi-simple eigenvalues which have constant mutliplicities.*

In the strictly hyperbolic case, the multiplicities are constant and equal to 1. Standard perturbation theory of matrices implies that eigenvalues of local constant multiplicity are smooth (real analytic) as well as the corresponding eigenprojectors. Therefore:

Lemma 2.3.10. *Hyperbolic systems with constant multiplicities, and in particular strictly hyperbolic systems, are smoothly diaganalizable and therefore strongly hyperbolic.*

2.3.5 Existence and uniqueness for strongly hyperbolic systems

Applying Theorem 2.1.9 immediately implies the following result.

Theorem 2.3.11. *If* (2.3.1) *is strongly hyperbolic, in particular if it is hyperbolic symmetrizable, then for all* $h \in H^s(\mathbb{R}^d)$ *and* $f \in L^1([0,T]; H^s(\mathbb{R}^d))$*,* (2.3.4) *defines* $u \in C^0([0,T]; H^s(\mathbb{R}^d)$ *which satisfies* (2.3.1) *and the estimates*

$$\big\|u(t)\big\|_{H^s} \leq C\big\|h\big\|_{H^s} + C\int_0^t \big\|f(t')\big\|_{H^s} dt'. \tag{2.3.17}$$

2.4 Higher order systems

The analysis of Section 1 can be applied to all systems with constant coefficients. We briefly study two examples.

2.4.1 Systems of Schrödinger equations

Extending (2.2.4), consider a $N \times N$ system

$$\partial_t u - i \sum_{j,k} A_{j,k} \partial_{x_j} \partial_{x_k} u + \sum_j B_j \partial_{x_j} u = f, \qquad u_{|t=0} = h. \tag{2.4.1}$$

On the Fourier side, it reads:

$$\partial_t \hat{u} + iP(\xi)\hat{u} = \hat{f}, \qquad \hat{u}_{|t=0} = \hat{h} \tag{2.4.2}$$

with

$$P(\xi) := \sum_{j,k} \xi_j \xi_k A_{j,k} + \sum_j \xi_j B_j := A(\xi) + B(\xi). \tag{2.4.3}$$

The Assumption 2.1.8 is satisfied when there are C and γ such that for $t > 0$ and $\xi \in \mathbb{R}^d$:

$$\big|e^{-itP(\xi)}\big| \leq Ce^{\gamma t}. \tag{2.4.4}$$

Case $1:\ B = 0$. Then $P(\xi) = A(\xi)$ is homogeneous of degree 2 and the discussion can be reduced to the sphere $\{|\xi| = 1\}$. Again, a necessary and sufficient condition is that the eigenvalues of $A(\xi)$ are real, semi-simple and the eigenprojectors are uniformly bounded.

Case 2 : $B \neq 0$. The discussion of (2.4.4) is much more delicate since the first order perturbation B can induce perturbations of order $O(|\xi|)$ in the spectrum of A. For instance, in the scalar case ($N = 1$), $P(\xi) = A(\xi) + B(\xi) \in \mathbb{C}$ and a necessary and sufficient condition for (2.4.4) is that for all ξ, $A(\xi)$ *and* $B(\xi)$ are real.

When $N \geq 2$, a sufficient condition is that A *and* B are real symmetric (or self-adjoint), since then $e^{itP(\xi)}$ is unitary. In the general case, $|\xi|$ large, the spectrum of $P(\xi)$ is a perturbation of the spectrum of $A(\xi)$ and therefore a necessary condition is that the eigenvalues of $A(\xi)$ must be real. Suppose the eigenvalues of that $A(\xi)$ have constant multiplicity so that $A(\xi)$ is smoothly diagonalizable:

$$A(\xi) = \sum \lambda_j(\xi)\Pi_j(\xi) \tag{2.4.5}$$

where the distinct eigenvalue λ_j are smooth and homogeneous of degree 2 and the eigenprojectors Π_j are smooth and homogeneous of degree 0. Then, for ξ large, one can block diagonalize P : with

$$\Omega = \mathrm{Id} + \sum_{j \neq k} (\lambda_k - \lambda_j)^{-1} \Pi_j B \Pi_k = \mathrm{Id} + O(|\xi|^{-1}) \tag{2.4.6}$$

there holds

$$\Omega^{-1} P \Omega = \sum \lambda_j \Pi_j + \Pi_j B \Pi_j + O(1). \tag{2.4.7}$$

Therefore, for (2.4.4) to be valid, it is necessary and sufficient that for all j:

i) λ_j is real,

ii) $e^{it\Pi_j B \Pi_j}$ is bounded.

This discussion is made in more details in Part III and extended to systems with variable coefficients.

2.4.2 Elasticity

Consider a second order system

$$\partial_t^2 u - \sum_{j,k} A_{j,k} \partial_{x_j} \partial_{x_k} u = f, \qquad u_{|t=0} = h_0, \quad \partial_t u_{|t=0} = h_1. \tag{2.4.8}$$

On the Fourier side, it reads

$$\partial_t^2 \hat{u} + A(\xi)\hat{u} = \hat{f}, \qquad \hat{u}_{|t=0} = \hat{h}_0, \quad \partial_t \hat{u}_{|t=0} = \hat{h}_1. \tag{2.4.9}$$

with $A(\xi) := \sum \xi_j \xi_k A_{j,k}$. The analysis performed for the wave equation can is easily extended to the case where

Assumption 2.4.1. *For all $\xi \neq 0$, $A(\xi)$ has only real and positive eigenvalues $\lambda_j(\xi)$ and the eigenprojectors $\Pi_j(\xi)$ are uniformly bounded.*

For instance, this assumption is satisfied when $A(\xi)$ is self-adjoint definite positive for all ξ.

Under the assumption above, the square root

$$K(\xi) = \sum \lambda_j^{\frac{1}{2}}(\xi)\Pi_j(\xi)$$

is well defined and homogeneous of degree 1 in ξ. Moreover, $K(\xi)$ is invertible for $\xi \neq 0$ and K^{-1} is homogeneous of degree -1. The matrices

$$e^{itK(\xi)} = \sum e^{it\lambda_j(\xi)}\Pi_j(\xi)$$

uniformly bounded as well as the matrices $\cos(tK)$ and $\sim(tK)$. The solution of (2.4.9) is

$$\begin{aligned} \hat{u}(t,\xi) = \cos(tK(\xi))\hat{h}_0(\xi) + \sin(tK(\xi))K^{-1}(\xi)\hat{h}_1(\xi) \\ + \int_0^t \sin((t-t')K(\xi))K^{-1}(\xi)\hat{f}(t',\xi)dt'. \end{aligned} \tag{2.4.10}$$

This implies that that Cauchy problem for (2.4.8) is well posed in Sobolev spaces, in the spirit of Theorem 2.2.2.

Chapter 3

The Method of Symmetrizers

In this chapter, we present the general principles of the method of proof of energy estimates using multipliers. To illustrate the method, we apply it to case of constant coefficient hyperbolic systems, where the computations are simple, explicit and exact. Of course, in this case, the estimates for the solutions were already present in the previous chapter, obtained from explicit representations of the solutions using Fourier synthesis. These explicit formula do not extend (easily) to systems with variable coefficients, while the method of symmetrizers does. In this respect, this chapter is an introduction to Part III. The constant coefficients computations will serve as a guideline in the more complicated case of equations with variable coefficients, to construct symbols, which we will transform into operators using the calculus of Part II.

3.1 The method

Consider an equation or a system

$$\begin{cases} \partial_t u + A(t)u = f & \text{on } [0,T] \times \mathbb{R}^d, \\ u_{|t=0} = h & \text{on } \mathbb{R}^d. \end{cases} \tag{3.1.1}$$

where $A(t) = A(t, x, \partial_x)$ is a differential operator in x depending on time:

$$A(t, x, \partial_x) = \sum A_\alpha(t, x) \partial_x^\alpha u \tag{3.1.2}$$

The "method" applies to abstract Cauchy problems (3.1.1) where u and f are functions of time $t \in [0, \infty[$ with values in some Hilbert space $\mathcal{H}$ and $A(t)$ is a C^1 family of (possibly unbounded) operators defined on $\mathcal{D}$, dense subspace of $\mathcal{H}$. Typically, for our applications $\mathcal{H} = L^2(\mathbb{R}^d; \mathbb{C}^N)$ and $\mathcal{D} = H^m(\mathbb{R}^d)$ where m is the order of A.

Definition 3.1.1. *A symmetrizer is a family of C^1 functions $t \mapsto S(t)$ with values in the space of bounded operators in $\mathcal{H}$ such that there are real numbers $C \ge c > 0$, C_1 and λ such that*

$$\forall t \in [0,T], \quad S(t) = S(t)^* \quad \text{and} \quad c\,\mathrm{Id} \le S(t) \le C\,\mathrm{Id}, \tag{3.1.3}$$

$$\forall t \in [0,T], \quad \big|\partial_t S(t)\big| \le C_1, \tag{3.1.4}$$

$$\forall t \in [0,T], \quad \mathrm{Re}\, S(t)A(t) \ge -\lambda \mathrm{Id}. \tag{3.1.5}$$

In (3.1.3), $S^*(t)$ is the adjoint operator of $S(t)$. The notation $\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$ is used in (3.1.5) for the real part of an operator T. When T is unbounded, the meaning of $\mathrm{Re}\, T \ge \lambda$, is that all $u \in \mathcal{D}$ belongs to the domain of T and satisfies

$$\mathrm{Re}\, \big(Tu, u\big)_{\mathcal{H}} \ge -\lambda |u|^2, \tag{3.1.6}$$

where $(\,\cdot\,)_{\mathcal{H}}$ is the scalar product in $\mathcal{H}$. The property (3.1.5) has to be understood in this sense.

For $u \in C^1([0,T]; \mathcal{D})$, taking the scalar product of Su with the equation (3.1.1) yields:

$$\frac{d}{dt}\big(S(t)u(t), u(t)\big)_{\mathcal{H}} + \big(K(t)u(t), u(t)\big)_{\mathcal{H}} = 2\mathrm{Re}\,\big(S(t)f(t), u(t)\big)_{\mathcal{H}}, \tag{3.1.7}$$

where

$$K(t) = 2\mathrm{Re}\, S(t)A(t) - \partial_t S(t). \tag{3.1.8}$$

Denote by

$$\mathcal{E}(u(t)) = \big(S(t)u(t), u(t)\big)_{\mathcal{H}} \tag{3.1.9}$$

the *energy* of u at time t. By (3.1.3),

$$c\big\|u(t)\big\|^2_{\mathcal{H}} \le \mathcal{E}(u(t)) \le C\big\|u(t)\big\|^2_{\mathcal{H}}. \tag{3.1.10}$$

Moreover, by Cauchy-Schwarz inequality,

$$\mathrm{Re}\, \big(S(t)f(t), u(t)\big)_{\mathcal{H}} \le \mathcal{E}(u(t))^{\frac{1}{2}}\ \mathcal{E}(f(t))^{\frac{1}{2}}. \tag{3.1.11}$$

Similarly, by (3.1.5) and (3.1.4), there holds

$$\big(K(t)u(t), u(t)\big)_{\mathcal{H}} \ge -(C_1 + 2\lambda)\big\|u(t)\big\|^2_{\mathcal{H}} \ge -2\gamma \mathcal{E}(u(t)) \tag{3.1.12}$$

with

$$\gamma = \frac{1}{2c}(C_1 + 2\lambda). \tag{3.1.13}$$

Therefore:

$$\frac{d}{dt}\mathcal{E}(u(t)) \leq 2\gamma\mathcal{E}(u(t)) + 2\mathcal{E}(u(t))^{\frac{1}{2}}\ \mathcal{E}(f(t))^{\frac{1}{2}}. \tag{3.1.14}$$

Hence:

Lemma 3.1.2. *If S is a symmetrizer, then for all $u \in C^1_0([0,\infty[;\mathcal{H}) \cap C^0([0,\infty[;\mathcal{D})$ there holds*

$$\mathcal{E}(u(t))^{\frac{1}{2}} \leq e^{\gamma t}\mathcal{E}(u(0))^{\frac{1}{2}} + \int_0^t e^{\gamma(t-t')}\mathcal{E}(f(t'))^{\frac{1}{2}}dt' \tag{3.1.15}$$

where $f(t) = \partial_t u + A(t)u(t)$ and γ is given by (3.1.13)*.*

3.2 The constant coefficients case

3.2.1 Fourier multipliers

Consider a constant coefficient system

$$\begin{cases} \partial_t u + A(\partial_x)u = f & \text{on } [0,T]\times\mathbb{R}^d, \\ u_{|t=0} = h & \text{on } \mathbb{R}^d, \end{cases} \tag{3.2.1}$$

where

$$A(\partial_x) = \sum A_\alpha \partial_x^\alpha u \tag{3.2.2}$$

After spatial Fourier transform, the system reads

$$\begin{cases} \partial_t \hat{u} + A(i\xi)\hat{u} = \hat{f} & \text{on } [0,T]\times\mathbb{R}^d, \\ \hat{u}_{|t=0} = \hat{h} & \text{on } \mathbb{R}^d. \end{cases} \tag{3.2.3}$$

It is natural to look for symmerizers that are defined on the Fourier side.

Proposition 3.2.1. *Suppose that $p(\xi)$ is a measurable function on $\mathbb{R}^d$ such that for some $m \in \mathbb{R}$:*

$$(1+|\xi|^2)^{-\frac{m}{2}}p \in L^\infty(\mathbb{R}^d). \tag{3.2.4}$$

Then the operator

$$p(D_x)u := \mathscr{F}^{-1}\big(p\hat{u}\big) \tag{3.2.5}$$

is bounded from $H^s(\mathbb{R}^d)$ *to* $H^{s-m}(\mathbb{R}^d)$ *for all* s *and*

$$\big\|p(D_x)u\big\|_{H^{s-m}} \leq \big\|(1+|\xi|^2)^{-\frac{m}{2}}p\big\|_{L^\infty}\big\|u\big\|_{H^s}. \tag{3.2.6}$$

This extends immediately to vector valued functions and matrices p.

Definition 3.2.2. *A function* p *satisfying* (3.2.4) *is called a Fourier multiplier of order* $\leq m$ *and* $p(D_x)$ *is the operator of symbol* $p(\xi)$*.*

The definition (3.2.5) and Plancherel's theorem immediately imply the following.

Proposition 3.2.3 (Calculus for Fourier Multipliers)**.** *i) If* p *and* q *are Fourier multipliers, then*

$$p(D_x) \circ q(D_x) = (pq)(D_x). \tag{3.2.7}$$

ii) Denoting by $p^*(\xi)$ *the adjoint of the matrix* $p(\xi)$*, then the adjoint of* $p(D_x)$ *in* L^2 *is*

$$\big(p(D_x)\big)^* = p^*(D_x). \tag{3.2.8}$$

iii) If p *is a self adjoint matrix of Fourier multipliers, then* $p(D_x) \geq c\mathrm{Id}$ *in the sense of self adjoints operators in* L^2 *if and only if for all* ξ $p(\xi) \geq c\mathrm{Id}$ *in the sense of self-adjoint matrices.*

An immediate corollary of this calculus is the following

Proposition 3.2.4. *For* $S(D_x)$ *to be a symmetrizer of* (3.2.1) *it is necessary and sufficient that there exist constants* $C \geq c > 0$ *and* λ *such that*

$$\forall \xi \in \mathbb{R}^d, \quad S(\xi) = S(\xi)^* \quad \text{and} \quad c\,\mathrm{Id} \leq S(\xi) \leq C\,\mathrm{Id}\,, \tag{3.2.9}$$

$$\forall \xi \in \mathbb{R}^d, \quad \mathrm{Re}\, S(\xi)A(i\xi) \geq -\lambda \mathrm{Id}\,. \tag{3.2.10}$$

3.2.2 The first order case

Consider a $N \times N$ first order system:

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u \tag{3.2.11}$$

Theorem 3.2.5. *i) The system L has a symmetrizer $S(D_x)$ associated to a Fourier multiplier $S(\xi)$ homogeneous of degree 0 if and only if it is strongly hyperbolic.*

ii) The symbol S can be taken constant independent of ξ if and only if the system is symmetrizable.

Proof. If $S(\xi)$ satisfies (3.2.10), then by homogeneity and evenness

$$\text{Im}\,\big(S(\xi)A(\xi)\big) = 0 \qquad \text{i.e.} \qquad \big(S(\xi)A(\xi)\big)^* = S(\xi)A(\xi). \tag{3.2.12}$$

This means that $A(\xi)$ is self-adjoint with respect to the scalar product associated to $S(\xi)$. Thus the eigenvalues of $A(\xi)$ are real and semi-simple and the eigenprojectors are of norm ≤ 1 in this hermitian structure. By (3.2.9), they are uniformly bounded.

Conversely, if L is strongly hyperbolic then

$$A(\xi) = \sum \lambda_j(\xi)\Pi_j(\xi), \qquad \text{Id} = \sum \Pi_j(\xi) \tag{3.2.13}$$

where the λ_j are real and the Π_j are uniformly bounded projectors such that $\Pi_j\Pi_k = \delta_{j,k}\Pi_j$. The matrix

$$S(\xi) = \sum \Pi_j^*\Pi_j(\xi) \tag{3.2.14}$$

is self-adjoint and

$$S(\xi)A(\xi) = \sum_{j,k} \lambda_j(\xi)\Pi_k^*\Pi_k(\xi)\Pi_j(\xi) = \sum_j \lambda_j(\xi)\Pi_j^*(\xi)\Pi_j(\xi) \tag{3.2.15}$$

is self-adjoint. Moreover, since $|u| \le \sum |\Pi_j u|$,

$$\frac{1}{N}|u|^2 \le \sum |\Pi_j u|^2 = Su\cdot u \le N \max |\Pi_j|^2 |u|^2$$

thus S satisfies (3.2.9).

Property *ii)* is just a rephrasing of the definition of symmetrizability. □

3.3 Hyperbolic symmetric systems

In this section, we briefly discuss the case of symmetric hyperbolic systems, as a first application of the method of symmetrizers.

3.3.1 Assumptions

Consider a first order $N \times N$ linear system:

$$\partial_t u + \sum_{j=1}^{d} A_j(t,x)\partial_{x_j} u + B(t,x)u = f, \qquad u_{|t=0} = h, \tag{3.3.1}$$

Assumption 3.3.1. *The coefficients of the matrices A_j are of class C^1, bounded with bounded derivatives on $[0,T] \times \mathbb{R}^d$. The coefficients of B are bounded on $[0,T] \times \mathbb{R}^d$.*

Assumption 3.3.2. *There is a matrix $S(t,x)$ such that*

- its coefficients are of class C^1, bounded with bounded derivatives on $[0,T] \times \mathbb{R}^d$.

- for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $S(t,x)$ is self-adjoint and positive definite,

- S^{-1} is bounded on $[0,T] \times \mathbb{R}^d$,

- for all j and all $(t,x) \in [0,T] \times \mathbb{R}^d$, the matrices SA_j are self-adjoint.

S is called a symmetrizer.

Maxwell equations or equations of acoustics presented in Chapter 1 are examples of symmetric systems.

Until the end of this section, the Assumptions 3.3.1 and 3.3.2 are supposed to be satisfied.

3.3.2 Existence and uniqueness

We give here without proof the classical existence and uniqueness theorem concerning hyperbolic-symmetric systems (see [Fr1, Fr2]). For a proof, we refer to Chapter 6.

Theorem 3.3.3. *For $h \in L^2(\mathbb{R}^d)$ and t $f \in L^1([0,T]; L^2(\mathbb{R}^d))$, the Cauchy problem (3.3.1) has a unique solution $u \in C^0([0,T]; L^2(\mathbb{R}^d)$.*

Moreover, there is C independent of the data f and h, such that for all $t \in [0,T]$:

$$\big\| u(t) \big\|_{L^2} \le C \big\| h \big\|_{L^2} + C \int_0^t \big\| f(t') \big\|_{L^2} dt'. \tag{3.3.2}$$

3.3.3 Energy estimates

We use the method of symmetrizers to prove *energy estimates* for the (smooth) solutions of (3.3.1). As shown in Chapter 6, these estimates are the key point in the proof of Theorem 3.3.3.

For simplicity, we assume that the coefficients of the equations and of the symmetrizer are *real* and we restrict ourselves to *real* valued solutions. We denote by $u \cdot v$ the scalar product of u and v taken in $\mathbb{R}^N$.

In many applications, for a function u with values in $\mathbb{R}^N$, the $S(t,x)u(t,x)\big)\cdot u(t,x)$ often corresponds to an energy density. It satisfies:

Lemma 3.3.4. *For $u \in C^1$, there holds*

$$\partial_t(Su \cdot u) + \sum_{j=1}^{d} \partial_{x_j}(SA_j u \cdot u) = 2Sf \cdot u + 2Ku \cdot u \tag{3.3.3}$$

with

$$f = \partial_t u + \sum_{j=1}^{d} SA_j \partial_{x_j} u + Bu \tag{3.3.4}$$

$$K = \partial_t S + \sum_{j=1}^{d} \partial_{x_j}(SA_j) - SB. \tag{3.3.5}$$

Proof. The chain rule implies

$$\partial(Gu \cdot u) = (G\partial u) \cdot u + Gu \cdot (\partial u) + (\partial G)u \cdot u.$$

When G is real symmetric, the first two terms are equal. Using this identity for $G = S$, $\partial = \partial_t$ and $G = SA_j$, $\partial = \partial_{x_j}$ implies (3.3.4). ☐

Consider a domain $\Omega \subset [0,T] \times \mathbb{R}^d$. Denote by Ω_τ the truncated subdomain $\Omega_\tau = \Omega \cap \{t \leq \tau\}$ and by ω_t the slices $\omega_t = \{x : (t,x) \in \Omega\}$. The boundary of Ω_τ is made of the upper and lower slices ω_τ and ω_0 and of a lateral boundary Σ_τ.

For instance when Ω is a cone

$$\Omega = \big\{(t,x) : \ 0 \leq t \leq T, \ |x| + \mu t \leq R\big\}, \tag{3.3.6}$$

for $t \leq \min\{T, R/\mu\}$ the slices are the balls

$$\omega_t = \big\{x : |x| \leq R - \mu t\big\}, \tag{3.3.7}$$

and the lateral boundary is

$$\Sigma_\tau = \big\{(t,x) : \ 0 \le t \le \tau, \ |x| + \mu t = R\big\} \tag{3.3.8}$$

Integrating (3.3.3) over Ω_τ, Green's formula implies that

Lemma 3.3.5. *With notations as in Lemma* 3.3.4*, there holds*

$$\begin{aligned}\int_{\omega_\tau} Su\cdot u dx = \int_{\omega_0} Su\cdot u dx - \int_{\Sigma_\tau} Gu\cdot u\, d\sigma \\ + 2\int_{\Omega_\tau} (Sf\cdot u + Ku\cdot u) dt dx\end{aligned} \tag{3.3.9}$$

where $d\sigma$ is the surface element on Σ_τ and for $(t,x)\in\Sigma_\tau$,

$$G = \nu_0 S + \sum_{j=1}^n \nu_j S A_j. \tag{3.3.10}$$

where $\nu = (\nu_0, \nu_1, \dots, \nu_n)$ is the unit outward normal to Σ_τ.

In the computation above, one can take $\Omega = [0,T]\times\mathbb{R}^d$. In this case there is no lateral boundary Σ, but integrability conditions at infinity are required. They are satisfied in particular when u has a compact support in x. Therefore

Lemma 3.3.6. *For u of class C^1 with compact support in $[0,T]\times\mathbb{R}^d$, there holds*

$$\int_{\mathbb{R}^d} (Su\cdot u)_{|t=\tau} dx = \int_{\mathbb{R}^d} (Su\cdot u)_{|t=0} dx + 2\int_{[0,\tau]\times\mathbb{R}^d} (Sf\cdot u + Ku\cdot u) dt dx \tag{3.3.11}$$

This is indeed a particular case of the identity (3.1.7) integrated between 0 and τ. Introduce the global energy at time t of u:

$$\mathcal{E}(t;u) = \int_{\mathbb{R}^d} S(t,x) u(t,x)\cdot u(t,x) dx \tag{3.3.12}$$

Theorem 3.3.7. *There is a constant C such that for all u of class C^1 with compact support in $[0,T]\times\mathbb{R}^d$, there holds for $t\in[0,T]$:*

$$\mathcal{E}(t;u)^{\frac12} \le e^{Ct}\mathcal{E}(0;u)^{\frac12} + \int_0^t e^{C(t-t')}\mathcal{E}(t';f) dt'. \tag{3.3.13}$$

Proof. This is indeed a particular case of Lemma 3.1.2.

Since $S(t,x)$ is symmetric definite positive, bounded with bounded inverse there are constants $m > 0$ and $M \ge m$ such that

$$\text{(3.3.14)} \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad \forall h \in \mathbb{R}^N : \qquad m|h|^2 \le S(t,x)h \cdot h \le M|h|^2.$$

Therefore:

$$\text{(3.3.15)} \qquad \forall t \in [0,T] \times \mathbb{R}^d : \qquad m \big\| u(t) \big\|^2_{L^2} \le \mathcal{E}(t;u) \le M \big\| u(t) \big\|^2_{L^2}.$$

The Cauchy-Schwarz inequality implies that for all (t,x) and all vectors k and h :

$$\text{(3.3.16)} \qquad \big| S(t,x)h \cdot k \big| \le \big(S(t,x)h \cdot h \big)^{\frac{1}{2}} \big(S(t,x)k \cdot k \big)^{\frac{1}{2}}.$$

Taking $h = f(t,x)$ and $k = u(t,x)$, integrating x and using the Cauchy-Schwarz inequality for the integral implies that

$$\text{(3.3.17)} \qquad \forall t \in [0,T] : \qquad \int_{\mathbb{R}^d} (Sf \cdot u)(t,x)dx \le \mathcal{E}(t;f)^{\frac{1}{2}} \mathcal{E}(t;u)^{\frac{1}{2}}.$$

The assumptions imply that the matrix K defined in (3.3.5) is bounded. With (3.3.14), we conclude that there is a constant C such that for all $u \in C^0$ with compact support in $[0,T] \times \mathbb{R}^d$, the following estimate is satisfied:

$$\text{(3.3.18)} \qquad \forall t \in [0,T] : \qquad \int_{\mathbb{R}^d} (Ku \cdot u)(t,x)dx \le C\mathcal{E}(t;u).$$

Introduce $\varphi(t) = \mathcal{E}(t;u)^{\frac{1}{2}}$ and $\psi(t) = \mathcal{E}(t;f)^{\frac{1}{2}}$. The identity (3.3.11) and the estimates above imply that

$$\text{(3.3.19)} \qquad \varphi(t)^2 \le \varphi(0)^2 + 2\int_0^t \psi(t')\varphi(t')dt' + C\int_0^t \varphi(t')^2 dt'.$$

This integral inequality implies

$$\text{(3.3.20)} \qquad \varphi(t) \le e^{\frac{1}{2}Ct}\varphi(0) + \int_0^t e^{\frac{1}{2}C(t-t')}\psi(t')dt'$$

that is (3.3.13) with the constant $\frac{1}{2}C$. $\square$

Proof of (3.3.20). Let $y(t)$ denote the right hand side of (3.3.19). This is a nonnegative nondecreasing function of t. It is differentiable and

$$y'(t) = 2\psi(t)\varphi(t) + C\varphi(t)^2 \le 2\psi(t)\sqrt{y(t)} + Cy(t).$$

Thus $z(t) = e^{-Ct} y(t)$ satisfies

$$z'(t) \leq 2e^{-Ct}\, \psi(t) \sqrt{y(t)} = 2e^{-\frac{1}{2}Ct} \psi(t) \sqrt{z(t)}.$$

Therefore

$$\sqrt{z(t)} \leq \sqrt{z(0)} + \int_0^t e^{-\frac{1}{2}Ct'}\, \psi(t') dt'.$$

and

$$\varphi(t) \leq \sqrt{y(t)} \leq e^{\frac{1}{2}Ct} \Big(\sqrt{z(0)} + \int_0^t e^{-\frac{1}{2}Ct'}\, \psi(t') dt' \Big).$$

□

Next we turn to local estimates. The key remark is that the boundary integral over Σ_τ occurring in (3.3.9) can be made ≥ 0 by choosing properly the domain Ω. For instance:

Lemma 3.3.8. *Consider a cone Ω as in* (3.3.6)*. There is μ_0 such that for $\mu \geq \mu_0$ the symmetric boundary matrix G given in* (3.3.10) *is nonnegative.*

Proof. The outward unit normal at $(t, x) \in \Sigma$ is

$$\nu_0 = \frac{\mu}{\sqrt{1 + \mu^2}}, \quad \nu_j = \frac{1}{\sqrt{1 + \mu^2}} \frac{x_j}{|x|}.$$

Since S is uniformly definite positive and the SA_j are uniformly bounded, it is clear that $G = \nu_0 S + \sum_{j=1}^n \nu_j S A_j$ is nonnegative if μ is large enough. □

Assuming that $\mu \geq \mu_0$, the equality (3.3.9) implies the inequality

$$\int_{\omega_\tau} Su \cdot u dx \leq \int_{\omega_0} Su \cdot u dx + 2 \int_{\Omega_\tau} (Sf \cdot u + Ku \cdot u) dt dx \tag{3.3.21}$$

From here, one can repeat the proof of Theorem 3.3.7 and show that the local energy

$$\mathcal{E}_\Omega(t, u) = \int_{\omega_t} S(t, x) u(t, x) \cdot u(t, x)\, dx \tag{3.3.22}$$

satisfies

Theorem 3.3.9. *There is a constant C such that if Ω is the cone* (3.3.6) *with $\mu \geq \mu_0$ and u is of class C^1 there holds for $t \in [0, T]$:*

$$\mathcal{E}_\Omega(t; u)^{\frac{1}{2}} \leq e^{Ct} \mathcal{E}_\Omega(0; u)^{\frac{1}{2}} + \int_0^t e^{C(t - t')} \mathcal{E}_\Omega(t'; f) dt'. \tag{3.3.23}$$

In particular, if $f = 0$ on Ω and $u_{|t=0} = 0$ on ω_0, then $u = 0$ on Ω. This is the key step in the proof of *local* uniqueness and finite speed of propagation for hyperbolic symmetric systems.

Part II

The Para-Differential Calculus

Chapter 4

Pseudo-differential operators

This chapter is devoted to a quick presentation of the language of pseudo-differential operators, in the most classical sense. The important points in this chapter are

- the notion of operators and symbol, with the exact calculus when the symbol are in the Schwartz class;
- the notion of symbols of type $(1, 1)$ as this is the class where the para-differential calculus takes place;
- Stein's theorem for the action of operators of type (1, 1);
- the crucial notion of spectral condition for the symbols as this is the key feature of the para-differential symbols;
- the extension of Stein's theorem to such operators.

One key idea, coming from harmonic analysis, is to use in a systematic way the Littlewood-Paley decomposition of functions and operators. In particular, we start with a characterization of classical function spaces using the Littlewood-Paley analysis.

4.1 Fourier analysis of functional spaces

Notations: Recall that $\mathscr{F}$ denotes the Fourier transform acting on temperate distributions $\mathscr{S}'(\mathbb{R}^d)$. We use its properties listed in Theorem 2.1.2.

The *spectrum* of u is the support of $\hat{u}$.

Fourier multipliers are defined by the formula

$$p(D_x)u = \mathscr{F}^{-1}(p\,\mathscr{F}u) \tag{4.1.1}$$

provided that the multiplication by p is defined at least form $\mathscr{S}$ to $\mathscr{S}'$. $p(D_x)$ is the operator associated to the *symbol* $p(\xi)$.

Function spaces. Recall the following definitions.

Definition 4.1.1. *For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is the space of distributions $u \in \mathscr{S}'(\mathbb{R}^d)$ such that their Fourier transform is locally integrable and*

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int (1 + |\xi|^2)^s \big|\hat{u}(\xi)\big|^2 d\xi < +\infty. \tag{4.1.2}$$

Definition 4.1.2 (Lipschitz and Hölder spaces)**.** *i) For $m \in \mathbb{N}$ we denote by $W^{m,\infty}(\mathbb{R}^d)$ the space of functions $u \in L^\infty(\mathbb{R}^d)$ such that all their derivatives $\partial^\alpha u$ of order $|\alpha| \leq m$ belong to $L^\infty(\mathbb{R}^d)$.*

ii) For $\mu \in]0, 1[$, we denote by $W^{\mu,\infty}(\mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d$ such that

$$\big[u\big]_\mu := \sup \frac{|u(x) - u(y)|}{|x - y|^\mu} < +\infty. \tag{4.1.3}$$

iii) for $\mu > 0$, $\mu \notin \mathbb{N}$, denoting by $[\mu]$ the greatest integer $< \mu$, the space $W^{\mu,\infty}(\mathbb{R}^d)$ is the space of functions in $W^{[\mu],\infty}(\mathbb{R}^d)$ such that their derivatives $\partial^\alpha u$ of order $|\alpha| = [\mu]$ belong to $W^{\mu-[\mu],\infty}$.

iv) For $m \in \mathbb{N}$, $C^m_b(\mathbb{R}^d)$ denotes the space of functions in $W^{m,\infty}(\mathbb{R}^d)$ such that all their derivatives of order $\leq m$ are continuous.

Remarks 4.1.3. $W^{1,\infty}$ is the space of bounded and Lipschitz functions on $\mathbb{R}^d$, that is which satisfy (4.1.3) with $\mu = 1$.

When $\mu \notin \mathbb{N}$, the notations $W^{\mu,\infty}$ is not quite standard for Hölder spaces. However, it is convenient for us to use the unified notations $W^{\mu,\infty}$ for $\mu \in \mathbb{N}$ and $\mu \notin \mathbb{N}$.

The definition of spaces $W^{\mu,\infty}$ will be extended to $\mu < 0$ ($\mu \notin \mathbb{Z}$) after Proposition 4.1.16

All these spaces are equipped with the obvious norms.

4.1.1 Smoothing and approximation.

We list here several useful lemmas concerning the approximation and the regularization of functions.

We consider in this section *families* of functions $\chi_\lambda \in \mathscr{S}(\mathbb{R}^d)$ such that

$$\begin{cases} \forall (\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d, \qquad \exists C_{\alpha,\beta} : \\ \forall \lambda \geq 1,\ \forall \xi \in \mathbb{R}^d : \quad \big|\xi^\alpha \partial^\beta_\xi \chi_\lambda(\xi)\big| \leq C_{\alpha,\beta} \lambda^{|\alpha| - |\beta|}. \end{cases} \tag{4.1.4}$$

Example 4.1.4. Take $\chi \in \mathscr{S}$ and $\chi_\lambda(\xi) = \chi(\lambda^{-1}\xi)$.

Remark 4.1.5. The condition (4.1.4) is equivalent to the condition that the family $\tilde{\chi}_\lambda(\xi) := \chi_\lambda(\lambda\xi)$ is bounded in $\mathscr{S}$. In the example above $\tilde{\chi}_\lambda = \chi$ is fixed.

Let $\varphi_\lambda = \mathscr{F}^{-1}\chi_\lambda \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\chi_\lambda(D_x)u(x) = \int u(x-y)\varphi_\lambda(y)dy. \tag{4.1.5}$$

The remark above implies that $\varphi_\lambda(x) = \lambda^d\tilde{\varphi}_\lambda(\lambda x)$ where $\tilde{\varphi}_\lambda = \mathscr{F}^{-1}\tilde{\chi}_\lambda$ is bounded in $\mathscr{S}$. Therefore, there are constants $C_{\alpha,\beta}$ such that

$$\int \left|x^\alpha \partial_x^\beta \varphi_\lambda(x)\right| dx \le C_{\alpha,\beta}\ \lambda^{|\beta|-|\alpha|}. \tag{4.1.6}$$

Lemma 4.1.6. *Suppose that the family $\{\chi_\lambda\}$ satisfies (4.1.4). For all $\alpha \in \mathbb{N}^d$, there is a constant C_α such that for all $\lambda > 0$, the operators $\partial_x^\alpha \chi_\lambda(D_x)$ are bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $1 \le p \le q \le +\infty$ with norm less than or equal to $C_\alpha \lambda^{|\alpha|+\frac{d}{p}-\frac{d}{q}}$.*

Proof. $\partial_x^\alpha \chi_\lambda(D_x)u$ is the convolution operator by $\partial_x^\alpha\varphi_\lambda(x) = \lambda^{d+|\alpha|}(\partial_x^\alpha\tilde{\varphi}_\lambda(\lambda x)$. Since $\tilde{\varphi}_\lambda$ is bounded in $\mathscr{S}$,

$$\left\|\partial_x^\alpha \varphi_\lambda\right\|_{L^r} \le C_\alpha \lambda^{|\alpha|+d(1-\frac{1}{r})}$$

and the lemma follows from Young's inequality. □

Corollary 4.1.7 (Bernstein's inequalities)**.** *Suppose that $a \in L^p(\mathbb{R}^d)$ has its spectrum contained in the ball $\{|\xi| \le \lambda\}$. Then $a \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$ and $q \ge p$, there is $C_{\alpha,p,q}$ (independent of λ) such that*

$$\|\partial_x^\alpha a\|_{L^q(\mathbb{R}^d)} \le C_{\alpha,p,q}\lambda^{|\alpha|+\frac{d}{p}-\frac{d}{q}}\|a\|_{L^p(\mathbb{R}^d)}\,. \tag{4.1.7}$$

In particular,

$$\|\partial_x^\alpha a\|_{L^p(\mathbb{R}^d)} \le C_\alpha\lambda^{|\alpha|}\|a\|_{L^p(\mathbb{R}^d)}, \qquad p=2,\ p=\infty, \tag{4.1.8}$$

$$\|a\|_{L^\infty(\mathbb{R}^d)} \le C\lambda^{\frac{d}{2}}\|a\|_{L^2(\mathbb{R}^d)} \tag{4.1.9}$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ supported in $\{|\xi| \le 2\}$ and equal to 1 for $|\xi| \le 1$. Then $\hat{a} = \chi_\lambda\hat{a}$ where $\chi_\lambda(\xi) = \chi(\lambda^{-1}\xi)$. Thus $a = \chi_\lambda(D_x)a$ and (4.1.7) follows from the previous Lemma. □

Lemma 4.1.8. *Suppose that the family* $\{\chi_\lambda\}$ *satisfies* (4.1.4) *and that each* χ_λ *vanishes on a neighborhood of the origin. For* $\mu > 0$*, there is a constant* C_μ *such that :*

for all $u \in W^{\mu,\infty}(\mathbb{R}^d)$*, one has the following estimate :*

$$\big\|\chi_\lambda(D_x)u\big\|_{L^\infty} \leq C_\mu \big\|u\big\|_{W^{\mu,\infty}} \lambda^{-\mu}. \tag{4.1.10}$$

Proof. Note that the estimate follows from Lemma 4.1.6 when $\lambda \leq 1$.

Since χ_λ vanishes in a neighborhood of the origin, there holds

$$\int y^\alpha \varphi_\lambda(y)dy = D_\xi^\alpha \chi_\lambda(0) = 0.$$

Therefore, (4.1.5) implies that

$$\chi_\lambda(D_x)u(x) = \int \Big(u(x-y) - \sum_{|\alpha|<\mu} \frac{(-y)^\alpha}{\alpha!}\partial_x^\alpha u(x)\Big)\varphi_\lambda(y)dy. \tag{4.1.11}$$

When $\mu \leq 1$ we use that

$$\big|u(x-y) - u(x)\big| \leq C\big\|u\big\|_{W^{\mu,\infty}} |y|^\mu. \tag{4.1.12}$$

When $\mu > 1$, we use Taylor's formula at order $n = \mu - 1$ when $\mu \in \mathbb{N}$ and at order $n = [\mu]$ when $\mu \notin \mathbb{N}$. It implies that

$$\begin{aligned} & u(x-y) - \sum_{|\alpha|<\mu} \frac{(-y)^\alpha}{\alpha!}\partial_x^\alpha u(x) \\ & = \sum_{|\alpha|=n} \frac{(-y)^\alpha n}{\alpha!} \int_0^1 (1-t)^{n-1} \big(\partial_x^\alpha u(x-ty) - \partial_x^\alpha u(x)\big)dt \end{aligned} \tag{4.1.13}$$

Thus the the integrand in (4.1.11) is $O\big(|y|^\mu \; \varphi_\lambda(y)\|u\|_{W^{\mu,\infty}}\big)$ and therefore

$$\big|\psi_\lambda(D_x)u(x)\big| \leq C\big\|u\big\|_{W^{\mu,\infty}} \int |y|^\mu |\varphi_\lambda(y)|dy. \tag{4.1.14}$$

Together with (4.1.6) this implies (4.1.10). □

Lemma 4.1.9. *Suppose that the family* $\{\chi_\lambda\}$ *satisfies* (4.1.4) *and that each* χ_λ *is equal to* 1 *on a neighborhood of the origin. For* $\mu > 0$*, there is a constant* C_μ *such that for all* $u \in W^{\mu,\infty}$ *:*

$$\big\|u - \chi_\lambda(D_x)u\big\|_{L^\infty} \leq C\big\|u\big\|_{W^{\mu,\infty}} \lambda^{-\mu} \tag{4.1.15}$$

Proof. The proof is quite similar. The inverse Fourier transform φ_λ now satisfy

$$\int \varphi_\lambda(y)du = 1 \qquad \text{and} \qquad \int y^\alpha \varphi_\lambda(y)dy = 0 \quad \text{when } |\alpha| > 0.$$

Therefore,

$$\chi_\lambda(D_x)u(x) - u(x) = \int \Big(u(x-y) - \sum_{|\alpha|<\mu} \frac{(-y)^\alpha}{\alpha!}\partial_x^\alpha u(x)\Big)\varphi_\lambda(y)dy. \tag{4.1.16}$$

The end of the proof is identical. □

Corollary 4.1.10. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $a \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$, one has the following estimate :*

$$\|a\|_{L^\infty} \leq C\|a\|_{W^{\mu,\infty}}\, \lambda^{-\mu} \tag{4.1.17}$$

Proof. $a = a - \chi(\lambda^{-1}D_x)a$ if χ is equal to 1 near the origin is supported in the ball of radius 1. □

4.1.2 The Littlewood-Paley decomposition in H^s.

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfy $0 \leq \chi \leq 1$ and

$$\chi(\xi)\ = 1 \quad \text{for } |\xi|\ \leq 1.1\,, \quad \chi(\xi)\ = 0 \quad \text{for } |\xi|\ \geq 1.9\,. \tag{4.1.18}$$

For $k \in \mathbb{Z}$, let

$$\chi_k(\xi) = \chi(2^{-k}\xi), \qquad \psi_k = \chi_k - \chi_{k-1}. \tag{4.1.19}$$

Introduce the operators acting on $\mathscr{S}'$:

$$S_k u = \mathscr{F}^{-1}\big(\chi(2^{-k}\xi)\hat{u}(\xi)\big) \tag{4.1.20}$$

and $\Delta_k = S_k - S_{k-1}$. In particular

$$u = S_0 u + \sum_{k=1}^{\infty} \Delta_k u \tag{4.1.21}$$

Proposition 4.1.11. *Consider $s \in \mathbb{R}$. A temperate distribution u belongs to $H^s(\mathbb{R}^d)$ if and only if*

i) $u_0 := S_0 u \in L^2(\mathbb{R}^d)$ and for all $k > 0$, $u_k := \Delta_k u \in L^2(\mathbb{R}^d)$

ii) the sequence $\delta_k = 2^{ks}\|u_k\|_{L^2(\mathbb{R}^d)}$ belongs to $\ell^2(\mathbb{N})$.

Moreover, there is a constant C, independent of u, such that

$$\frac{1}{C}\|u\|^2_{H^s} \le \Big(\sum_k \delta_k^2\Big)^{1/2} \le C\|u\|^2_{H^s} \tag{4.1.22}$$

Proof. In the frequency space there holds

$$\widehat{u} = \sum_{k=1}^{\infty} \widehat{u_k} \tag{4.1.23}$$

Let $\theta_0 = \chi_0$ and $\theta_k = \psi_k$ for $k \ge 1$. Because $0 \le \theta_k \le 1$, there holds

$$\sum \big|\widehat{u_k}(\xi)\big|^2 = \sum \theta_k^2(\xi)\big|\hat{u}(\xi)\big|^2 \le \sum \theta_k(\xi)\big|\hat{u}(\xi)\big|^2 = \big|\hat{u}(\xi)\big|^2$$

On the other hand, every ξ belongs at most to the support of 3 functions θ_k. Therefore

$$\big|\hat{u}(\xi)\big|^2 = \Big|\sum \widehat{u_k}(\xi)\Big|^2 \le 3\sum \big|\widehat{u}_k(\xi)\big|^2.$$

Summing up, we have proved that

$$\sum \big|\widehat{u}_k(\xi)\big|^2 \le \big|\hat{u}(\xi)\big|^2 \le 3\sum \big|\widehat{u}_k(\xi)\big|^2. \tag{4.1.24}$$

Multiplying by $(1+|\xi|^2)^s$, integrating over $\mathbb{R}^d$, and noticing that

$$\frac{1}{4}\,2^{2k} \le 1 + |\xi|^2 \le 4\,2^{2k} \qquad \text{on the support of } \ \theta_k, \tag{4.1.25}$$

the proposition follows. □

Proposition 4.1.12. *Consider $s \in \mathbb{R}$ and $R > 0$. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R}^d)$such that*

i) the spectrum of u_0 is contained in the ball $\{|\xi| \le R\}$ and for $k > 0$ the spectrum of u_k is contained in $\big\{\frac{1}{R}2^k \le |\xi| \le R2^k\big\}$.

ii) the sequence $\delta_k = 2^{ks}\|u_k\|_{L^2(\mathbb{R}^d)}$ belongs to $\ell^2(\mathbb{N})$.

Then $u = \sum u_k$ belongs to $H^s(\mathbb{R}^d)$ and there is a constant C, independent of the sequence such that

$$\|u\|^2_{H^s} \le C\Big(\sum_k \delta_k^2\Big)^{1/2}$$

When $s > 0$, it is sufficient to assume that the spectrum of u_k is contained in the ball $\big\{|\xi| \le R2^k\big\}$.

Proof. Define the θ_j as in the previous proof. By Lemma 4.1.6,

$$\|\theta_j(D_x)u_k\|_{L^2} \leq C\|u_k\|_{L^2} \leq C2^{-ks}\delta_k.$$

Moreover, the spectral assumption in $i)$ implies that $\theta_j(D_x)u_k = 0$ if $|k-j| \geq a = \ln(2R)/\ln 2$. Thus

$$\|\theta_j(D_x)u\|_{L^2} \leq C2^{-js}\tilde{\delta}_j, \qquad \tilde{\delta}_j = \sum_{|k-j|\leq a} 2^{s(j-k)}\delta_k \tag{4.1.26}$$

When the spectrum of u_k is contained in the ball $\{|\xi| \leq R2^k\}$, then $\theta_j(D_x)u_k = 0$ when $j \geq k+a$. Thus the estimate in (4.1.26) is satisfied with

$$\tilde{\delta}_j = \sum_{k\geq j-a} 2^{s(j-k)}\delta_k.$$

When $s > 0$, we see that this sequence $(\tilde{\delta}_j)$ still belongs to ℓ^2 as a consequence of Young's inequality for the convolution of sequences, one in ℓ^2, the other in ℓ^1. □

We will also use another version where the spectral localization is replaced by estimates which mimic this localization.

Proposition 4.1.13. *Let $0 < s$ and let n be an integer, $n > s$. There is a constant C such that :*

for all sequence $(f_k)_{k\geq 0}$ in $H^n(\mathbb{R}^d)$ satisfying for all $\alpha \in \mathbb{N}^d$ of length $|\alpha| \leq n$

$$\|\partial_x^\alpha f_k\|_{L^2(\mathbb{R}^d)} \leq 2^{k(|\alpha|-s)}\varepsilon_k, \qquad \textit{with } (\varepsilon_k) \in \ell^2, \tag{4.1.27}$$

the sum $f = \sum f_k$ belongs to $H^s(\mathbb{R}^d)$ and

$$\|f\|^2_{H^s(\mathbb{R}^d)} \leq C\sum_{k=0}^{\infty}\varepsilon_k^2. \tag{4.1.28}$$

Proof. Since $s > 0$, the series $\sum f_k$ converge in $L^2(\mathbb{R}^d)$ and $\widehat{f} = \sum \widehat{f}_k$. There holds

$$\|\theta_j(D_x)f_k\|_{L^2} \leq C\|f_k\|_{L^2} \leq C2^{-ks}\varepsilon_k,$$
$$\|\theta_j(D_x)f_k\|_{L^2} \leq C2^{-nj}\|f_k\|_{H^n} \leq C2^{-ks}2^{n(k-j)}\varepsilon_k.$$

We use the first estimate when $j \leq k$ and the second when $j > k$. Therefore,

$$\|\theta_j(D_x)f\|_{L^2} \leq C2^{-js}(\varepsilon'_j + \varepsilon''_j) \tag{4.1.29}$$

with

$$\varepsilon'_j = \sum_{k\geq j} 2^{(j-k)s}\varepsilon_k, \qquad \varepsilon''_j = \sum_{k<j} 2^{(n-s)(k-j)}\varepsilon_k.$$

Because $s < n$, (ε''_j) belongs to ℓ^2 and because $s > 0$, (ε'_j) belongs to ℓ^2.Moreover, their norms are dominated by the ℓ^2 norm of the sequence (ε_k). □

The restriction $s > 0$ can be dropped when the f_k satisfy an appropriate spectral condition.

Proposition 4.1.14. *Let $s \in \mathbb{R}$, $\kappa > 0$ and let $n > s$ be an integer. There is a constant C such that :*

for all sequence $(f_k)_{k\geq 0}$ in $H^n(\mathbb{R}^d)$ satisfying (4.1.27) *and*

$$\text{supp}\hat{f}_k \subset \{\xi \,:\, 1 + |\xi| \geq \kappa 2^k\}, \tag{4.1.30}$$

$f = \sum f_k$ belongs to $H^s(\mathbb{R}^d)$ and satisfies (4.1.28)

Proof. The spectral condition implies that there is N such that $\theta_j(D_x) f_k = 0$ when $j < k - N$. Therefore the estimate (4.1.29) is satisfied with ε'_j now defined by

$$\varepsilon'_j = \sum_{k=j}^{j+N} 2^{(j-k)s}\varepsilon_k.$$

Noticing that this sequence (ε'_j) belongs to ℓ^2 when $(\varepsilon_k) \in \ell^2$, implies the proposition. □

The estimates of $\|\Delta_k u\|_{L^2}$ can be combined with Lemma 4.1.6. In particular, for $u \in H^s(\mathbb{R}^d)$ there holds

$$\big\|\Delta_k u\big\|_{L^\infty} \leq \varepsilon_k 2^{-k(s-\frac{d}{2})} \tag{4.1.31}$$

with $\{\varepsilon_k\} \in \ell^2$. Summing in k immediately implies the following results.

Proposition 4.1.15 (Sobolev embeddings)**.** *i) If $s > \frac{d}{2}$, then $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and there is a constant C such that for $u \in H^s(\mathbb{R}^d)$:*

$$\big\|u\big\|_{L^\infty} \leq C\big\|u\big\|_{H^s}. \tag{4.1.32}$$

ii) If $s < \frac{d}{2}$, there is a constant C such that for $u \in H^s(\mathbb{R}^d)$ and all k

$$\big\|S_k u\big\|_{L^\infty} \leq \varepsilon_k 2^{k(\frac{d}{2}-s)}, \qquad \sum_{k=0}^{\infty} \varepsilon_k^2 \leq C\big\|u\big\|^2_{H^s}. \tag{4.1.33}$$

4.1.3 The Littlewood-Paley decomposition in Hölder spaces.

Proposition 4.1.16. *Consider $\mu > 0$, $\mu \notin \mathbb{N}$. A temperate distribution u belongs to $W^{\mu,\infty}(\mathbb{R}^d)$ if and only if*

i) $u_0 := S_0 u \in L^\infty$ and for all $k > 0$, $u_k := \Delta_k u \in L^\infty(\mathbb{R}^d)$

ii) the sequence $\delta_k = 2^{k\mu}\|u_k\|_{L^\infty(\mathbb{R}^d)}$ belongs to $\ell^\infty(\mathbb{N})$.

Moreover, there is a constant C, independent of u, such that

$$\frac{1}{C}\big\|u\big\|_{W^{\mu,\infty}} \leq \sup_k \delta_k \leq C\big\|u\big\|_{W^{\mu,\infty}} \tag{4.1.34}$$

Proof. By Lemma 4.1.6

$$\big\|S_0 u\big\|_{L^\infty} \leq C\big\|u\big\|_{L^\infty}.$$

The estimate of $\Delta_k u$ is a particular case of Lemma 4.1.10.

Conversely, if $\|u_k\|_{L^\infty} \leq C2^{-k\mu}$, then Lemma 4.1.7 implies that for $|\alpha| < \mu$, $\|\partial^\alpha u_k\|_{L^\infty} \leq C2^{-k(\mu-|\alpha|)}$. This shows that the series $\sum \partial^\alpha u_k$ converges uniformly an thus $u = \sum u_k \in C_b^{[\mu]}$. Next, we use that for $|\alpha| = [\mu]$

$$\begin{aligned}
&|\partial^\alpha u_k(x) - \partial^\alpha u_k(y)| \leq C2^{-k(\mu-[\mu])},\\
&|\partial^\alpha u_k(x) - \partial^\alpha u_k(y)| \leq |x-y|\|\nabla\partial^\alpha u_k\|_{L^\infty}(y)| \leq C|x-y|2^{k(1-\mu+[\mu])}.
\end{aligned}$$

We use the first estimate when $2^{-k} \leq |x-y|$ and the second when $|x-y| < 2^{-k}$. Using that $0 < \mu - [\mu] < 1$, the estimates sums in k and we obtain that

$$\big|\partial^\alpha u_k(x) - \partial^\alpha u_k(y)\big| \leq C'|x-y|^{(\mu-[\mu])},$$

which proves that $u \in C_b^\mu$. $\square$

For $\mu < 0$, $\mu \notin \mathbb{Z}$, we can take the properties i) and ii) as a *definition* of the space $W^{\mu,\infty}$:

Definition 4.1.17. *Consider $\mu < 0$, $\mu \notin \mathbb{Z}$. A temperate distribution u belongs to $W^{\mu,\infty}(\mathbb{R}^d)$ if and only if*

i) $u_0 := S_0 u \in L^\infty$ and for all $k > 0$, $u_k := \Delta_k u \in L^\infty(\mathbb{R}^d)$

ii) the sequence $\delta_k = 2^{k\mu}\|u_k\|_{L^\infty(\mathbb{R}^d)}$ belongs to $\ell^\infty(\mathbb{N})$.

Using Lemma 4.1.6, one can check that the space does not depend on the particular choice of the cut-off function defining the Littlewood-Paley decomposition. There are results analogous to Propositions 4.1.12 and 4.1.13, but be omit them.

Remark 4.1.18. The characterization above *does not* extend to the case $\mu \in \mathbb{N}$. However, the second inequality in (4.1.34) is still true. The next proposition collects several useful results concerning the spaces $W^{m,\infty}(\mathbb{R}^d)$, $m \in \mathbb{N}$.

Proposition 4.1.19. *There is a constant C such that :*

i) for all $u \in L^\infty$ and all $k \in \mathbb{N}$, one has

$$\|S_k u\|_{L^\infty} \leq C\|u\|_{L^\infty} ,$$

ii) for all $u \in W^{m,\infty}$ and all $k \in \mathbb{N}$, one has

$$\|\Delta_k u\|_{L^\infty} \leq C2^{-km}\|u\|_{W^{1,\infty}} , \quad \|u - S_k u\|_{L^\infty} \leq C2^{-km}\|u\|_{W^{1,\infty}} .$$

Proof. i) has already been stated in (4.1.8). The estimates of $\Delta_k u$ and $u - S_k u$ are particular cases of Lemmas 4.1.6 and 4.1.9. □

Finally, we quote the following estimates which will be useful later on.

Proposition 4.1.20. *Given a real number $r > 0$ and an integer $n \geq r$, there is a constant C such that for all k, $u \in W^{r,\infty}$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| = n$:*

$$\left\|\partial_x^\alpha S_k u\right\|_{L^\infty} \leq C2^{k(n-r)}\left\|u\right\|_{W^{r,\infty}}$$

Proof. When $r \notin \mathbb{N}$, we write $S_k u = S_0 u + \sum 1 \leq j \leq k\Delta_j u$ and use the estimates

$$\left\|\partial_x^\alpha S_0 u\right\|_{L^\infty} \leq C\left\|u\right\|_{L^\infty}, \tag{4.1.35}$$

$$\left\|\partial_x^\alpha \Delta_j u\right\|_{L^\infty} \leq C2^{j(n-r)}\left\|u\right\|_{W^{r,\infty}}.$$

which follows directly from Proposition 4.1.16 and the Bernstein's inequalities (4.1.8).

When $r \in \mathbb{N}$ and $|\alpha| = n \geq r$, there are α' and α'' such that $\alpha = \alpha' + \alpha''$ and $|\alpha'| = r$. Then, from the Bernstein's inequalities (4.1.8) and Proposition 4.1.19, we see that

$$\begin{aligned}\left\|\partial_x^\alpha S_k u\right\|_{L^\infty} &\leq C2^{n-r}\left\|S_k \partial_x^{\alpha''} u\right\|_{L^\infty} \\ &\leq C'2^{(n-r)}\left\|\partial_x^{\alpha''} u\right\|_{L^\infty} \leq C2^{k(n-r)}\left\|u\right\|_{W^{r,\infty}}.\end{aligned}$$

□

4.2 The general framework of pseudo-differential operators

4.2.1 Introduction

Recall that the Fourier multiplier $p(D_x)$ is defined by (4.1.1). It is defined as soon as the multiplication by p acts from $\mathscr{S}$ to $\mathscr{S}'$. The main properties of Fourier mulitpliers are that

- $p(D_x) \circ q(D_x) = (pq)(D_x)$,
- $\big(p(D_x)\big)^* = (\overline{p})(D_x)$,
- if $p \geq 0$, then $p(D_x)$ is nonnegative as an operator,

The goal of pseudo-differential calculus is to extend the definition (4.1.1) to symbols $p(x, \xi)$, by the following formula:

$$\big(p(x, D_x)u\big)(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} p(x, \xi)\widehat{u}(\xi)d\xi\,, \tag{4.2.1}$$

and to show that the properties above remain true, not in an exact sense but up to remainder terms which are smoother.

4.2.2 Operators with symbols in the Schwartz class

As an introduction, we first study the case of operators defined by symbols in the Schwartz class. The results will be extended to more general symbols in the following sections.

For $p \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and $u \in \mathscr{S}(\mathbb{R}^d)$, $p(x, \xi)\hat{u}(\xi)$ and $e^{ix\cdot\xi}p(x, \xi)\widehat{u}(\xi)$ belong to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ so that the integral in (4.2.1) is convergent and defines a function in the Schwarz class $\mathscr{S}(\mathbb{R}^d)$. Substituting the definition of $\hat{u}$ yields the convergent integral

$$(2\pi)^{-d} \int e^{i(x-y)\cdot\xi} p(x, \xi)\, u(y) d\xi\, dy,$$

so that

$$\big(p(x, D_x)u\big)(x) = \int (\mathscr{F}_\xi^{-1}p)(x, x - y)\, u(y) dy\,, \tag{4.2.2}$$

where $\mathscr{F}_\xi^{-1}p \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ denotes the inverse Fourier transform of p with respect to the variables ξ. Thus the kernel $K(x, y) = (\mathscr{F}_\xi^{-1}p)(x, x - y)$ belongs to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and this clearly implies :

Lemma 4.2.1. *If $p \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, the operator $p(x, D_x)$ extends as a continuous operator from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$ and for u and v in $\mathscr{S}'(\mathbb{R}^d)$:*

$$\big\langle p(x,\partial_x)v, u\big\rangle_{\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^d} = \big\langle K, u\otimes v\big\rangle_{\mathscr{S}(\mathbb{R}^r\times\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^r\times\mathbb{R}^d)}. \tag{4.2.3}$$

Conversely, for any $K \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ the symbol $p = \mathscr{F}_z K(x, x-z)$, that is

$$p(x,\xi) = \int e^{-iz\cdot\xi} K(x, x-z) dz, \tag{4.2.4}$$

belongs to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Thus the theory of pseudo-differential operators with symbols in the Schwarz class $\mathscr{S}$ is nothing but the theory of operators with kernels in the Schwarz class.

On the Fourier side, for $u \in \mathscr{S}$ the Fourier transform of (4.2.1) is given by the absolutely convergent integral

$$(2\pi)^{-d} \int e^{ix\cdot(\xi-\eta)} p(x,\xi)\widehat{u}(\xi) d\xi\, dx,$$

and therefore

$$\mathscr{F}\big(p(x, D_x)u\big)(\eta) = (2\pi)^{-d} \int (\mathscr{F}_x p)(\eta-\xi,\xi)\widehat{u}(\xi) d\xi\,, \tag{4.2.5}$$

where $\mathscr{F}_x p \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ denotes the Fourier transform of p with respect to the variables x.

Lemma 4.2.2. *If p and q belong to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then $p(x, D_x) \circ q(x, D_x) = r(x, D_x)$ with*

$$r(x,\xi) = \frac{1}{(2\pi)^d} \int e^{iy\cdot\eta} p(x, \xi+\eta) q(x+y, \xi) dy d\eta. \tag{4.2.6}$$

Equivalently

$$r(x,\xi) \;:= e^{-ix\xi}\big(p(x, D_x)\tilde{q}_\xi\big)(x)\,, \quad \tilde{q}_\xi(\xi) := e^{ix\xi} q(x,\xi)\,. \tag{4.2.7}$$

Proof. By (4.2.5),

$$\begin{aligned} p(x, D_x) \circ q(x, D_x)u(x) &= \frac{1}{(2\pi)^{2d}} \int e^{ix\cdot\eta} p(x,\eta)(\mathscr{F}_x q)(\eta-\xi,\xi)\hat{u}(\xi) d\xi d\eta \\ &= \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} r(x,\xi)\hat{u}(\xi) d\xi \end{aligned}$$

with

$$r(x,\xi) = \frac{1}{(2\pi)^d} \int e^{ix\cdot(\eta-\xi)} p(x,\eta)(\mathscr{F}_x q)(\eta-\xi,\xi) d\eta$$
$$= \frac{1}{(2\pi)^d} \int e^{i(x-z)\cdot\zeta} p(x,\xi+\zeta) q(z,\xi) dz d\zeta.$$

The change of variables $z = x+y$ yields (4.2.6) while the change of variables $\zeta = \xi' - \xi$ yields (4.2.7). Note that all the integrals above are absolutely convergent and that $r \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$. □

We define $(p(x,D_x))^*$ as the adjoint of $p(x,D_x)$ acting in L^2, that is as the transposed of $p(x,\partial_x)$ for the anti-duality $\langle u, \overline{v}\rangle$:

$$\text{(4.2.8)} \qquad \big\langle \big(p(x,D_x)\big)^* u, \overline{v}\big\rangle_{\mathscr{S}\times\mathscr{S}'} = \big\langle u, \overline{p(x,D_x)v}\big\rangle_{\mathscr{S}'\times\mathscr{S}}.$$

For u and v in $\mathscr{S}$ this means that

$$\text{(4.2.9)} \qquad \int \big(p(x,D_x)\big)^* u, \overline{v}\; dx = \int u, \overline{p(x,D_x)v}\, dx.$$

Lemma 4.2.3. *If p belongs to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then $\big(p(x,D_x)\big)^* = r(x,D_x)$ with*

$$\text{(4.2.10)} \qquad r(x,\xi) = \frac{1}{(2\pi)^d} \int e^{-iy\cdot\eta}\; \overline{p(x+y,\xi+\eta)} dy d\eta$$

and

$$\text{(4.2.11)} \qquad (\mathscr{F}_x r)(\eta,\xi) = (\mathscr{F}_x \overline{p})(\eta,\xi+\eta).$$

Proof. By (4.2.2), $p(x,D_x)$ is defined by the kernel $K(x,y) = (\mathscr{F}_\xi^{-1} p)(x, x-y)$ which belongs to the Schwartz class. Its adjoint is defined by the kernel $K^*(x,y) = \overline{K(y,x)} = (\mathscr{F}_\xi^{-1}\overline{p})(y, x-y)$. By (4.2.4) it is associated to the symbol

$$r(x,\xi) = \int e^{-iz\cdot\xi} (\mathscr{F}_\xi^{-1}\overline{p})(x-z,z) dz.$$

Thus

$$r(x,\xi) = \frac{1}{(2\pi)^d} \int e^{iz\cdot(\eta-\xi)} \overline{p}(x-z,\eta) dz d\eta$$

and (4.2.10) follows. Similarly,

$$(\mathscr{F}_x r)(\eta,\xi) = \int e^{-i(z\cdot\xi + x\cdot\eta)} (\mathscr{F}_\xi^{-1}\overline{p})(x-z,z) dz$$
$$= \int e^{-iz\cdot(\xi+\eta)} (\mathscr{F}_\xi^{-1}\mathscr{F}_x \overline{p})(\eta,z) dz dx = (\mathscr{F}_x\overline{p})(\eta,\xi+\eta).$$

□

4.2.3 Pseudo-differential operators of type (1, 1)

Definition 4.2.4. *For $m \in \mathbb{R}$, $S^m_{1,1}$ is the space of functions p, C^∞ on $\mathbb{R}^d \times \mathbb{R}^d$ such that for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that*

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad \left| \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \right| \le C_{\alpha,\beta} \left(1 + |\xi|\right)^{m+|\alpha|-|\beta|}. \tag{4.2.12}$$

$S^m_{1,0}$ is the subspace of symbols p such that for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad \left| \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \right| \le C_{\alpha,\beta} \left(1 + |\xi|\right)^{m-|\beta|}. \tag{4.2.13}$$

The best constant in (4.2.12) and (4.2.13) define semi-norms, to that $S^m_{1,1}$ and $S^m_{1,0}$ are equipped with natural topologies. In particular, a family of symbols p_k is said to be bounded in $S^m_{1,1}$ [resp. $S^m_{1,0}$] if they satisfy the estimates (4.2.12) [resp. (4.2.13)] with constants $C_{\alpha,\beta}$ independent of k.

Examples 4.2.5. • Smooth homogeneous functions of degree m, $h(\xi)$, are symbols of degree m for $|\xi| \ge 1$. Thus $\chi(\xi) h(\xi) \in S^m_{1,0}$ if $\chi \in C^\infty(\mathbb{R}^d)$ is equal to 1 outside a ball and vanishes near the origin.

• If $\chi \in C_0^\infty(\mathbb{R}^d)$, then for all $\lambda \ge 1$, $\chi_\lambda(\xi) := \chi(\lambda^{-1}\xi)$ is a symbol of degree 0 and the family $\{\chi_\lambda\}$ is bounded in $S^m_{1,0}$.

For such symbols and for $u \in \mathscr{S}(\mathbb{R}^d)$, the integral in (4.2.1) converges and can be differentiated at any order. Multiplying it by x^α and integrating by parts, shows that the the integral is rapidly decreasing in x. Therefore:

Proposition 4.2.6. *For $p \in S^m_{1,1}$, the relation* (4.2.1) *defines $p(x, D_x)$ as a continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to itself.*

To make rigorous several computations below, we need to approximate symbols in the classes $S^m_{1,1}$ or $S^m_{1,0}$ by symbols in the Schwartz class. Of course, this cannot be done in the topology defined by the semi-norms associated to the estimates (4.2.12) or (4.2.13). Instead we use a weaker form.

Lemma 4.2.7. *Given $p \in S^m_{1,1}$, there are symbols $p_k \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

i) the family $\{p_k\}$ is bounded in $S^m_{1,1}$,

ii) $p_k \to p$ on compact subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

For any such family and for all $u \in \mathscr{S}(\mathbb{R}^d)$, $p_k(x, D_x)u \to p(x, D_x)u$ in $\mathscr{S}(\mathbb{R}^d)$.

Proof. Let

$$p_k(x,\xi) = \psi(2^{-k}x)\chi(2^{-k}\xi)p(x,\xi) \tag{4.2.14}$$

where $\psi \in \mathscr{S}$ with $\psi(0) = 0$ and $\chi \in C_0^\infty$ equal to 1 on the unit ball. Then the family p_k satisfies *i)* and *ii)* (Exercise).

If the family $\{p_k\}$ is bounded in $S^m_{1,1}$ and $u \in \mathscr{S}(\mathbb{R}^d)$, then the family $\{p_k(x,D_x)u\}$ is bounded in $\mathscr{S}$; moreover, *ii)* implies that if u has compact spectrum, then $p_k(x,D_x)u \to p(x,D_x)u$ on any given compact set Thus $p_k(x,D_x)u \to p(x,D_x)u$ in $\mathscr{S}$. By density of C_0^∞ in $\mathscr{S}$ and uniform bounds of the $p_k(x,D_x)$, the convergence holds for $u \in \mathscr{S}$. □

4.2.4 Spectral localization

Localization in the space of frequencies is a central argument in the analysis developed in this chapter. In particular, the action pseudo-differential operators on spectra is a key point.

Proposition 4.2.8. *If $p \in S^m_{1,1}$ and $u \in \mathscr{S}(\mathbb{R}^d)$ then the spectrum of $p(x,D_x)u$ is contained in the closure of the set*

$$\big\{\xi + \eta, \quad \xi \in \mathrm{supp}\hat{u}, \ (\eta,\xi) \in \mathrm{supp}\mathscr{F}_x p\big\}. \tag{4.2.15}$$

Proof. The formula (4.2.5) extends to symbols $p \in S^m_{1,1}$, in the sense of distributions: $v = p(x,D)u$ satisfies for all $\varphi \in \mathscr{S}$:

$$\big\langle \hat{v}, \varphi\big\rangle = (2\pi)^{-d}\big\langle \mathscr{F}_x p, \varphi(\eta+\xi)\hat{u}(\xi)\big\rangle$$

If φ vanishes on a neighborhood of the set (4.2.15), then $\varphi(\eta+\xi)\hat{u}(\xi)$ vanishes on neighborhood of the support of $\mathscr{F}_x p$ and the proposition follows. □

We now introduce important subclasses of $S^m_{1,1}$.

Definition 4.2.9. *Let A symbol $\sigma(x,\xi) \in S^m_{1,1}$ is said to satisfy the spectral condition if*

$$\exists \varepsilon < 1 : \qquad \mathscr{F}_x\sigma(\eta,\xi) = 0 \quad for\ |\eta| \geq \varepsilon(|\xi|+1). \tag{4.2.16}$$

The space of such symbols is denoted by Σ^m_0.

Remark 4.2.10. The Bernstein inequalities of Corollary 4.1.7 show that the estimates

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \Big|\partial^\beta_\xi \sigma(x,\xi)\Big| \leq C_\beta\,(1+|\xi|)^{m-|\beta|} \tag{4.2.17}$$

and the spectral property (4.2.16) are sufficient to imply that σ satisfies (4.2.12), thus that $\sigma \in S^m_{1,1}$.

Lemma 4.2.11. *For all $\sigma \in \Sigma_0^m$, there is a sequence of symbols $\sigma_n \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

i) the family $\{\sigma_n\}$ is bounded in $S^m_{1,1}$,

ii) the σ_n satisfy the spectral property (4.2.16) *for some $\varepsilon < 1$ independent of k,*

iii) $\sigma_n \to \sigma$ on compact subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. For instance, consider

$$\sigma_n(x,\xi) = \psi(2^{-n}x)\chi(2^{-n}\xi)p(x,\xi) \tag{4.2.18}$$

with $\chi \in C_0^\infty$ equal to 1 on the unit ball and $\psi \in \mathscr{S}$ such that $\psi(0) = 0$ and $\widehat{\psi}$ is supported in $\{|\eta| \le 1\}$.

The Fourier transform of $\psi_n(x) = \psi(2^{-n}x)$ is contained in the ball $\{|\eta| \le 2^{-n}\}$. Thus the support of $\mathscr{F}_x\sigma_n(\,\cdot\,,\xi)$ which is the convolution of $\widehat{\psi}_n$ and $\mathscr{F}_x\sigma(\,\cdot\,,\xi)$ is contained in the ball of radius $\varepsilon(1+|\xi|) + 2^{-n} \le \varepsilon'(1+|\xi|)$ if n is large enough and $\varepsilon' > \varepsilon$ is fixed. □

A key property of operators with symbols in Σ_0^m is that they do not extend too much the spectrum of functions. Proposition 4.2.8 immediately implies the following property:

Lemma 4.2.12. *If $p \in \Sigma_0^m$ satisfies the spectral property with parameter $\varepsilon < 1$ and if $f \in \mathscr{S}(\mathbb{R}^d)$ has compact spectrum, then the spectrum of $Pf = p(x,D_x)f$ is contained in the set*

$$\big\{\xi + \eta, \quad \xi \in \mathrm{supp}\hat{f},\ |\eta| \le \varepsilon(1+|\xi|)\big\} \tag{4.2.19}$$

In particular:

- if $\mathrm{supp}\hat{f} \subset \{|\xi| \le R\}$, then $\mathrm{supp}(\widehat{Pf}) \subset \{|\xi| \le \varepsilon + (1+\varepsilon)R\}$,

- if $\mathrm{supp}\hat{f} \subset \{|\xi| \ge R\}$, then $\mathrm{supp}(\widehat{Pf}) \subset \{|\xi| \ge (1-\varepsilon)R - \varepsilon\}$,

4.3 Action of pseudo-differential operators in Sobolev spaces

4.3.1 Stein's theorem for operators of type (1, 1)

Theorem 4.3.1 (E. Stein)**.** *If $s > 0$ and $\mu > 0$ with $\mu \notin \mathbb{N}$ and $\mu + m \notin \mathbb{N}$, then for all $p \in S^m_{1,1}$, $p(x,D_x)$, extends as a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and from $W^{m+\mu,\infty}(\mathbb{R}^d)$ to $W^{\mu,\infty}(\mathbb{R}^d)$.*

Consider first the case of Sobolev spaces H^s. We wan to prove that there is a constant C such that for all $u \in \mathscr{S}(\mathbb{R}^d)$:

$$\big\|p(x, D_x)u\big\|_{H^s} \le C\big\|u\big\|_{H^{s+m}}. \tag{4.3.1}$$

By Lemma 4.2.7, we see that it is sufficient to prove such an estimate when $p \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with a constant C which depends only on a finite number of semi-norms on $S^m_{1,1}$.

To prove this inequality, we split p into dyadic pieces and use the dyadic analysis of Sobolev spaces. For each dyadic piece, the key estimate follows from the next lemma.

Lemma 4.3.2. *There are constants C and C' such that :*
for all $\lambda > 0$ and $q \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$\operatorname{supp} q \subset \mathbb{R}^d \times \big\{|\xi| \le \lambda\big\}, \qquad M := \sup_{|\beta| \le n} \lambda^{|\beta|}\big\|\partial_\xi^\beta q\big\|_{L^\infty} < \infty,$$

with $\tilde{d} = [\frac{d}{2}] + 1$, then the function

$$Q(y) = \int e^{-iy\xi} q(y, \xi) d\xi$$

satisfies

$$\int (1 + |\lambda y|^2)^{\tilde{d}}\, |Q(y)|^2\, dy \le C\lambda^d M^2\,, \tag{4.3.2}$$

and

$$\big\|Q\big\|_{L^1(\mathbb{R}^d)} \le C' M. \tag{4.3.3}$$

Proof. For $|\alpha| \le \tilde{d}$, there holds

$$y^\alpha Q(y) = \int e^{-iy\cdot\xi} D_\xi^\alpha q(y, \xi) d\xi\,.$$

Hence, by Plancherel's theorem

$$\int \big|y^{2\alpha}\big|\, |Q(y)|^2\, dy \le C\lambda^{d-2|\alpha|} M^2\,.$$

Multiplying by $\lambda^{2|\alpha|}$ and summing in α, implies (4.3.2). Since $\tilde{d} > \frac{d}{2}$, the second estimate (4.3.3) follows. $\square$

We now start the proof of Theorem 4.3.1. Consider $p \in S^m_{1,1}$ and introduce the semi-norms

$$\mathcal{M}^m_n(p) := \sup_{|\alpha|\le n} \sup_{|\beta|\le [\frac{d}{2}]+1} \sup_{x,\xi} \big|(1+|\xi|)^{|\beta|-m-|\alpha|}\partial_x^\alpha\partial_\xi^\beta p(x,\xi)\big|, \tag{4.3.4}$$

To prove Theorem 4.3.1, we split p into dyadic pieces :

$$p(x,\xi) = p_0(x,\xi) + \sum_{k=1}^{\infty} p_k(x,\xi)\,, \tag{4.3.5}$$

using χ as in (4.1.18) and defining

$$p_0(x,\xi) = p(x,\xi)\chi(\xi)\,, \quad p_k(x,\xi) = p(x,\xi)\big(\chi(2^{-k}\xi)-\chi(2^{1-k}\xi)\big) \quad \mathit{for}\ k\ge 1\,.$$

Lemma 4.3.3. *For $k \ge 0$, $P_k = p_k(x,D_x)$ maps L^2 to H^∞. Moreover, for all $\alpha\in\mathbb{N}^n$, there is C_α such that for all $p\in S^m_{1,1}$, $k\ge 0$ and all $f\in L^2$:*

$$\|\partial_x^\alpha P_k f\|_{L^2} \le C_\alpha \mathcal{M}^m_{|\alpha|}(p)\|f\|_{L^2}2^{k(m+|\alpha|)}\,. \tag{4.3.6}$$

Proof. Since p_k is compactly supported in ξ, one immediately sees that $P_k f$ is given by the convergent integral:

$$P_k f(x) = \int P_k(x,y)f(y)dy$$

where the kernel $P_k(x,y)$ is given by the convergent integral

$$P_k(x,y) = (2\pi)^{-d}\int e^{i(x-y)\xi}p_k(y,\xi)d\xi.$$

Moreover, on the support of p_k, $1+|\xi|\approx 2^k$. Therefore Lemma 4.3.2 can be applied with $\lambda = 2^{k+1}$, implying that

$$\int (1+2^{2k}|x-y|^2)^{\tilde d}\,|P_k(x,y)|^2\,dy \le C2^{2m+kd}\big(\mathcal{M}^m_0(p)\big)^2\,.$$

Hence, for $f\in\mathscr{S}(\mathbb{R}^d)$, Cauchy-Schwarz inequality implies that

$$|P_k f(x)|^2 \le C\big(\mathcal{M}^m_0(p)\big)^2\int \frac{2^{2m+kd}|f(y)|^2}{(1+2^{2k}|x-y|^2)^{\tilde d}}dy \tag{4.3.7}$$

Since $\tilde d > d/2$, the integral $2^{kd}\displaystyle\int(1+2^{2k}|x-y|^2)^{-d}dx = C'$ is finite and independent of k. Therefore, with a new constant C independent k and p:

$$\big\|P_k f\big\|^2_{L^2} \le C\mathcal{M}^m_0(p)2^{km}\big\|f\big\|^2_{L^2}$$

This proves (4.3.6) for $\alpha = 0$.

The symbol of $\partial_x^\alpha P_k$ is $(i\xi + \partial_x)^\alpha p_k(x,\xi)$. Thus there is a similar estimate, involving the semi norms $\mathcal{M}_0^{m+|\alpha|}(\partial_x^\alpha p)$ and the lemma follows. □

End of the proof of Theorem 4.3.1 in the scale H^s. For $f \in \mathscr{S}(\mathbb{R}^d)$, consider the dyadic decomposition (4.1.21):

$$f = f_0 + \sum_{k=1}^{\infty} f_k \,,$$

Note that $\widehat{f_j} = 0$ on the support of $p_k(x,\cdot)$ if $|j-k| \geq 4$. Thus

$$P_k f = \sum_{|j-k|\leq 3} P_k f_j \,.$$

Therefore, Lemma 4.3.3 and Proposition 4.1.11 imply that :

$$\begin{aligned} \big\|\partial_x^\alpha P_k f\big\|_{L^2} &\leq C_\alpha \mathcal{M}_n^m(p) \sum_{|j-k|\leq 3} 2^{k(|\alpha|+m)} \big\|f_j\big\|_{L^2} \\ &\leq C'_\alpha \mathcal{M}_n^m(p) 2^{k(|\alpha|-s)} \varepsilon_k \end{aligned} \tag{4.3.8}$$

with

$$\sum_k \varepsilon_k^2 \leq \|f\|^2_{H^{s+m}} \,. \tag{4.3.9}$$

By Lemma 4.2.7, $Pf = \sum P_k f$ in $\mathscr{S}$. When $s > 0$, Proposition 4.1.13 applies to the series $\sum P_k f$, implying that $Pf \in H^s$ and that there is a constant C_s such that

$$\big\|p(x,D_x)f\big\|_{H^s} \leq C_s \mathcal{M}_n^m(p) \big\|f\big\|_{H^{s+m}}, \tag{4.3.10}$$

where $\mathcal{M}_n^m(p)$ is the semi-norm (4.3.4) and $n > s$ is an integer. This implies Theorem 4.3.1 in the scale of Sobolev spaces H^s. □

Proof of Theorem 4.3.1 in the scale $W^{\mu,\infty}$. The proof is quite similar and since we will not use this result we omit the details. The basic point it the following analogue of the estimates (4.3.6) :

$$\big\|P_k f\big\|_{L^\infty} \leq C \mathcal{M}_n^m(p) 2^{k(m+|\alpha|)} \big\|f\big\|_{L^\infty} \tag{4.3.11}$$

This estimate immediately follows from (4.3.3) applied to p_k. □

4.3.2 The case of symbols satisfying spectral conditions

For symbols in Σ_0^m the restrictions $s > 0$ and $\mu > 0$ in Theorem 4.3.1 can be relaxed. More generally:

Theorem 4.3.4. *Consider $m \in \mathbb{R}$ and $s \in \mathbb{R}$ [resp. $\mu \notin \mathbb{N}$ with $\mu + m \notin \mathbb{N}$] Consider next $p \in S_{1,1}^m$ which satisfies the following condition:*

$$(4.3.12) \qquad \exists \delta > 0 \ : \qquad \mathrm{supp} \mathscr{F}_x p \subset \big\{ (\eta, \xi) \ : \ 1 + |\xi + \eta| \geq \delta |\xi| \big\}.$$

Then $p(x, D_x)$, extends as a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and from $W^{m+\mu,\infty}(\mathbb{R}^d)$ to $W^{\mu,\infty}(\mathbb{R}^d)$.

Proof. Split the symbol $p = \sum p_k$ as in (4.3.5) and consider the dyadic decomposition $f = \sum f_j$ of $f \in \mathscr{S}$. Taking into account the spectral localization of p_k and f_j, there holds

$$P_k f = \sum_{|j-k| \leq 3} P_k f_j \, .$$

The estimates (4.3.8) and (4.3.9) for $g_k = P_k f$ are still true. The new point is that

$$(4.3.13) \qquad \mathrm{supp} \hat{g}_k \subset \big\{ \eta \ : \ 1 + |\eta| \geq \delta 2^{k-1} \big\}$$

so that Proposition 4.1.14 can be applied. This implies the theorem in the scale of spaces H^s.

The spectral localization (4.3.13) follows from Proposition 4.2.8 and the assumption (4.3.12) which implies that

$$\mathrm{supp} \mathscr{F}_x p_k \subset \big\{ (\eta, \xi) \ : \ 1 + |\xi + \eta| \geq \delta |\xi|, \ |\xi| \geq 2^{k-1} \big\}.$$

when $k \geq 1$. When $k = 0$, the inclusion (4.3.13) is trivial if $\delta \leq 1$, as we may assume.

The proof in Hölder spaces $W^{\mu,\infty}$ is similar. $\square$

This theorem applies in particular to symbols $p \in \Sigma_0^m$, since the spectral property (4.2.16) implies (4.3.12) with $\delta = 1 - \varepsilon$. Moreover, the spectral condition and Bernstein's inequalities imply that the semi-norm (4.3.4) $\mathcal{M}_n^m(p)$ are dominated by $\mathcal{M}_0^m(p)$. Thus :

Theorem 4.3.5. *Consider $m \in \mathbb{R}$ and $s \in \mathbb{R}$ [resp. $\mu \notin \mathbb{N}$ with $\mu + m \notin \mathbb{N}$] If $p \in \Sigma_0^m$, then $p(x, D_x)$, extends as a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and from $W^{m+\mu,\infty}(\mathbb{R}^d)$ to $W^{\mu,\infty}(\mathbb{R}^d)$.*

More precisely, for all $\varepsilon < 1$, there is a constant C_ε such that for all $p \in \Sigma_0^m$ satisfying the spectral condition with parameter ε and all $f \in H^{s+m}(\mathbb{R}^d)$, there holds

$$\big\| p(x, D_x) f \big\|_{H^s} \le C_\varepsilon \mathcal{M}_0^m(p) \big\| f \big\|_{H^{s+m}}, \tag{4.3.14}$$

where $\mathcal{M}_0^m(p)$ is the semi-norm (4.3.4).

Chapter 5

Para-Differential Operators

The paradifferential calculus in $\mathbb{R}^d$, was introduced by J.M.Bony [Bon] (see also [Mey], [Hör], [Tay], [Mé1]). The para-differential quantization is a way to associate an operator to a symbol which has a limited smoothness in x. The first point in this chapter is the definition of the quantization (Definition 5.1.14) and the action of para-differential operators in functional spaces. Next we discuss the special case of symbols independent of ξ which leads to the definition of para-products (Section 5.2). The main concern is to compare products and para-products, or to split products into para-products (the key idea from Littlewood-Paley decompositions). This has two consequences of fundamental importance for applications to PDE's in Part III : the para-linearization theorems (Theorems 5.2.4 and Theorems 5.2.8 5.2.9) which allow to replace a nonlinear equation by a para-diffential linear one, to the price of an acceptable error.

5.1 Definition of para-differential operators

We first introduce classes of symbols and next define the associated operators.

5.1.1 Symbols with limited spatial smoothness

We first introduce a general definition:

Definition 5.1.1 (Symbols). *Given $m \in \mathbb{R}$ and a Banach space $\mathscr{W} \subset \mathscr{S}'(\mathbb{R}^d)$, $\Gamma^m_{\mathscr{W}}$ denotes the space of distributions $a(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$, which are C^∞ with respect to ξ and such that for all $\alpha \in \mathbb{N}^d$ there is a constant*

C_α *such that*

$$\forall \xi \ : \quad \left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{\mathscr{W}} \leq C_\alpha \, (1 + |\xi|)^{m - |\alpha|} \,. \tag{5.1.1}$$

$\Sigma^m_{\mathscr{W}}$ *denotes the subclass of symbols* $\sigma \in \Gamma^m_{\mathscr{W}}$ *which satisfy the following spectral condition*

$$\exists \varepsilon < 1 : \qquad \mathscr{F}_x \sigma(\eta, \xi) = 0 \quad for \ |\eta| > \varepsilon(|\xi| + 1). \tag{5.1.2}$$

We will use this definition mainly for $\mathscr{W} = L^\infty(\mathbb{R}^d)$ or $\mathscr{W} = W^{\mu,\infty}(\mathbb{R}^d)$, and occasionally for $\mathscr{W} = H^s(\mathbb{R}^d)$. In the former case, to simplify the exposition, we use the following special notations:

Definition 5.1.2. *i)* Γ^m_0 *denotes the space of locally* L^∞ *functions* $a(x, \xi)$ *on* $\mathbb{R}^d \times \mathbb{R}^d$ *which are* C^∞ *with respect to* ξ *and such that for all* $\alpha \in \mathbb{N}^d$ *there is a constant* C_α *such that*

$$\forall (x, \xi) \ : \quad |\partial_\xi^\alpha a(x, \xi)| \leq C_\alpha \, (1 + |\xi|)^{m - |\alpha|} \,. \tag{5.1.3}$$

ii) More generally, for $r \geq 0$, Γ^m_r *denotes the space of symbols* $a \in \Gamma^m_0$ *such that for all* $\alpha \in \mathbb{N}^d$ *and all* $\xi \in \mathbb{R}^d$, *the function* $x \mapsto \partial_\xi^\alpha a(x, \xi)$ *belongs to* $W^{r,\infty}$ *and there is a constant* C_α

$$\forall \xi \ : \quad \left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{r,\infty}} \leq C_\alpha \, (1 + |\xi|)^{m - |\alpha|} \,. \tag{5.1.4}$$

iii) Σ^m_r *denotes the subclass of symbols* $\sigma \in \Gamma^m_r$ *which satisfy the spectral condition* (5.1.2).

The spaces $\Gamma^m_{\mathscr{W}}$ are equipped with the natural topology and the semi-norms defined by the best constants in (5.1.1). In particular, for $p \in \Gamma^m_r$, we use the notations

$$M^m_r(p; n) = \sup_{|\beta| \leq n} \sup_{\xi \in \mathbb{R}^d} \left\| (1 + |\xi|)^{|\beta| - m} \partial_\xi^\beta p(\cdot, \xi) \right\|_{W^{r,\infty}}. \tag{5.1.5}$$

Remark 5.1.3. When $\mathscr{W} \subset L^\infty$, $\Gamma^m_{\mathscr{W}} \subset \Gamma^m_0$ and $\Sigma^m_{\mathscr{W}} \subset \Sigma^m_0$. Moreover, by Remark 4.2.10, $\Sigma^m_0 \subset S^m_{1,1}$. More generally, the spectral condition implies that symbols in $\Sigma^m_{\mathscr{W}}$ are smooth in x too.

5.1.2 Smoothing symbols

The symbols in the classes Γ^m are not smooth with respect to the first variable x. The next step in the construction is to smooth this symbols by using an appropriate truncation on the Fourier side, which depend on the frequency variable ξ.

Definition 5.1.4. *An admissible cut-off function is a C^∞ function $\psi(\eta,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

1) there are ε_1 and ε_2 such that $0<\varepsilon_1<\varepsilon_2<1$ and

$$\left\{\begin{array}{ll}\psi(\eta,\xi) = 1 & \text{for } |\eta| \le \varepsilon_1(1+|\xi|) \\ \psi(\eta,\xi) = 0 & \text{for } |\eta| \ge \varepsilon_2(1+|\xi|)\,.\end{array}\right. \tag{5.1.6}$$

2) for all $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that

$$\forall(\eta,\xi)\ :\quad |\partial_\eta^\alpha\partial_\xi^\beta\psi(\eta,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}\,. \tag{5.1.7}$$

Example 5.1.5. Consider a cut-off function $\chi\in C_0^\infty(\mathbb{R}^d)$ as in Section 4, satisfying $0\le\chi\le 1$ and

$$\chi(\xi) = 1 \quad \text{for } |\xi| \le 1.1\,,\qquad \chi(\xi) = 0 \quad \text{for } |\xi| \ge 1.9\,. \tag{5.1.8}$$

Let :

$$\psi_N(\eta,\xi) = \sum_{k=0}^{+\infty}\chi_{k-N}(\eta)\varphi_k(\xi) \tag{5.1.9}$$

where

$$\chi_k(\xi) = \chi\big(2^{-k}\xi\big) \quad \text{for } k\in\mathbb{Z} \tag{5.1.10}$$

$$\varphi_0 = \chi_0 \quad \text{and} \quad \varphi_k = \chi_k - \chi_{k-1} \quad \text{for } k\ge 1. \tag{5.1.11}$$

Then, for $N\ge 3$, ψ_N is an admissible cut off function in the sense of the definition above.

Remark 5.1.6 (Exercise)**.** We leave to the reader to check that for all $\varepsilon<1$, there is an admissible cut-off function ψ such that $\psi=1$ on the set $\{(\eta,\xi)\,,\,|\eta|\le\varepsilon(1+|\xi|)\}$.

Lemma 5.1.7. *Let ψ be an admissible cut-off, and $G^\psi(\,\cdot\,,\xi)$ be the inverse Fourier transform of $\psi(\,\cdot\,,\xi)$. Then, for all $\alpha\in\mathbb{N}^d$, there is C_α such that*

$$\forall\alpha\in\mathbb{N}^d\,,\quad \forall\xi\in\mathbb{R}^d:\quad \big\|\partial_\xi^\alpha G^\psi(\,\cdot\,,\xi)\big\|_{L^1(\mathbb{R}^d)} \le C_\alpha(1+|\xi|)^{-|\alpha|}\,. \tag{5.1.12}$$

Proof. The estimates (5.1.7) and the support condition (5.1.6) imply that for all (α,β) there is a constant $C_{\alpha,\beta}$ such that for all (x,ξ):

$$\big|x^\alpha\partial_\xi^\beta G^\psi(x,\xi)\big| \le C_{\alpha,\beta}(1+|\xi|)^{d-|\alpha|-|\beta|}\,.$$

Thus

$$\big|\partial_\xi^\beta G^\psi(x,\xi)\big| \le C_\beta(1+|\xi|)^{-|\beta|}\frac{(1+|\xi|)^d}{\big(1+|x|(1+|\xi|)\big)^{d+1}}\,.$$

and (5.1.12) follows. ∎

For $a \in \Gamma_0^\mu$ define

$$\sigma_a^\psi(x,\xi) := \int G^\psi(x-y,\xi)\, a(y,\xi)\, dy, \tag{5.1.13}$$

that is

$$\sigma_a^\psi(\,\cdot\,,\xi) = \psi(D_x,\xi)\, a(\,\cdot\,,\xi), \tag{5.1.14}$$

or equivalently on the Fourier side in x,

$$\mathscr{F}_x \sigma_a^\psi = \psi\, \mathscr{F}_x a. \tag{5.1.15}$$

Proposition 5.1.8. *Let ψ be an admissible cut-off.*

i) For all $m \in \mathbb{R}$ and $r \geq 0$, the operator $a \mapsto \sigma_a^\psi$ is bounded from Γ_r^m to Σ_r^m. More precisely, for all $n \in \mathbb{N}$ there is C_n such that

$$M_r^m(\sigma_a^\psi; n) \leq C_n M_r^m(a; n). \tag{5.1.16}$$

ii) If $a \in \Gamma_r^m$ with $r > 0$, then $a - \sigma_a^\psi \in \Gamma_0^{m-r}$. Moreover, for all $n \in \mathbb{N}$ there is C_n such that

$$M_0^{m-r}(a - \sigma_a^\psi; n) \leq C_n M_r^m(a; n). \tag{5.1.17}$$

In particular, if ψ_1 and ψ_2 are two admissible cut-off functions, then for $r > 0$ and $a \in \Gamma_r^m$ the difference $\sigma_a^{\psi_1} - \sigma_a^{\psi_2}$ belongs to Σ_0^{m-r} and for all $n \in \mathbb{N}$ there is C_n such that

$$M_0^{m-r}(\sigma_a^{\psi_1} - \sigma_a^{\psi_2}; n) \leq C_n M_r^m(a; n). \tag{5.1.18}$$

Proof. For $\varphi \in L^1(\mathbb{R}^d)$ and $u \in W^{\mu,\infty}$, the convolution $\varphi * u$ belongs to $W^{\mu,\infty}$ and

$$\left\| \varphi * u \right\|_{W^{\mu,\infty}} \leq \left\| \varphi \right\|_{L^1} \left\| u \right\|_{W^{\mu,\infty}}.$$

This implies $i)$ with $n = 0$ (no derivative in ξ). The estimate of the ∂_ξ^β derivatives is similar, using the chain rule.

The second statement immediately follows from Lemma 4.1.9. $\square$

As pointed out in Remark 4.2.10, symbols in Σ_0^m belong to $S_{1,1}^m$ and are infinitely differentiable in x. This applies to symbols σ_a^ψ. More precisely, there holds

Proposition 5.1.9. *i) For $m \in \mathbb{R}$, $r \geq 0$, $\alpha \in \mathbb{N}^d$ of length $|\alpha| \leq r$ and $a \in \Gamma_r^m$*

$$\partial_x^\alpha \sigma_a^\psi = \sigma_{\partial_x^\alpha a}^\psi \in \Sigma_0^m. \tag{5.1.19}$$

ii) For $m \in \mathbb{R}$, $r \geq 0$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| \geq r$ the mapping $a \mapsto \partial_x^\alpha \sigma_a^\psi$ is bounded from Γ_r^m to $\Sigma_0^{m+|\alpha|-r}$. More precisely, for all $n \in \mathbb{N}$ there is C_n such that for all $a \in \Gamma_r^m$:

$$M_0^{m+|\alpha|-r}(\partial_x^\alpha \sigma_a^\psi; n) \leq C_n M_r^m(a; n). \tag{5.1.20}$$

Proof. The relation (5.1.19) immediately follows from the definition (5.1.13)

When $|\alpha| \geq r$, we note that the spectral property and (5.1.14) imply that

$$\mathscr{F}_x \sigma_a^\psi(\,\cdot\,, \xi) \;=\; \psi(D_x, \xi)\, S_k(D_x) a(\,\cdot\,, \xi) \quad \text{if } 2^k \geq (1 + |\xi|). \tag{5.1.21}$$

Thus, taking k such that $(1 + |\xi|) \leq 2^k \leq 2(1 + |\xi|)$, Lemma 4.1.6 and Proposition 4.1.20 imply that

$$\big\| \partial_x^\alpha \sigma_a^\psi(\,\cdot\,, \xi) \big\|_{L^\infty} \leq C 2^{k(n-r)} \big\| a(\,\cdot\,, \xi) \big\|_{W^{r,\infty}}.$$

The estimates of the ξ derivatives are similar and they imply that $\partial_x^\alpha \sigma_a^\psi \in \Sigma_0^{m+|\alpha|-r}$. □

For ξ derivatives, there is an approximate analogue of (5.1.19):

Proposition 5.1.10. *For $m \in \mathbb{R}$, $r \geq 0$, $\beta \in \mathbb{N}^d$ and $a \in \Gamma_r^m$*

$$\partial_\xi^\beta \sigma_a^\psi - \sigma_{\partial_\xi^\beta a}^\psi \in \Sigma_0^{m-|\beta|-r}. \tag{5.1.22}$$

Proof. By (5.1.14),

$$\partial_\xi^\beta \sigma_a^\psi - \sigma_{\partial_\xi^\beta a}^\psi \;=\; \sum_{0<\gamma\leq\beta} \binom{\beta}{\gamma} (\partial_\xi^\gamma \psi)(D_x, \xi)\, \partial_\xi^{\beta-\gamma} a(\,\cdot\,, \xi),$$

When $\gamma > 0$, (5.1.6) implies that $\partial_\xi^\gamma \psi(\eta, \xi) = 0$ when $|\eta| \geq \varepsilon_2(1 + |\xi|)$. Therefore Lemma 4.1.8 implies that for $\gamma > 0$

$$\big\| (\partial_\xi^\gamma \psi)(D_x, \xi)\, \partial_\xi^{\beta-\gamma} a(\cdot, \xi) \big\|_{L^\infty} \leq C(1 + |\xi|)^{-r} \big\| \partial_\xi^{\beta-\gamma} a(\cdot, \xi) \big\|_{W^{r,\infty}}.$$

This implies (5.1.22). □

Remark 5.1.11. More generally, the mapping $a \mapsto \sigma_a^\psi$ is bounded from $\Gamma^m_{\mathscr{W}}$ to $\Sigma^m_{\mathscr{W}}$, provided that the convolution is bounded from $L^1 \times \mathscr{W}$ to $\mathscr{W}$. This applies in particular when $\mathscr{W} = H^s$, $s \geq 0$. One can also apply Proposition 4.1.15 and obtain the following result.

Proposition 5.1.12. *If $a \in \Gamma^m_{H^s}$ with $s < \frac{d}{2}$, then $\sigma_a^\psi \in \Sigma_0^{m+\frac{d}{2}-s}$.*

Proof. Use again (5.1.21). Propositions 4.1.15 and Lemma 4.1.6 imply that

$$\left\|\sigma_a^\psi(\,\cdot\,,\xi)\right\|_{L^\infty} \leq C(1+|\xi|)^{\frac{d}{2}-s}\left\|a(\,\cdot\,,\xi)\right\|_{H^s}.$$

The estimates of the ξ derivatives are similar and the proposition follows. □

In the same vein, let $W^{-1,\infty}(\mathbb{R}^d)$ denotes the space of distributions u which can be written $u = u_0 + \sum \partial_{x_j} u_j$ with $u_j \in L^\infty(\mathbb{R}^d)$, equipped with the norm

$$\left\|u\right\|_{W^{-1,\infty}} = \inf \sum_j \left\|u_j\right\|_{L,\infty} \tag{5.1.23}$$

where the infimum is taken over all the decompositions $u = u_0 + \sum \partial_{x_j} u_j$. Then :

Proposition 5.1.13. *If $a \in \Gamma^m_{W^{-1,\infty}}$, then $\sigma_a^\psi \in \Sigma_0^{m+1}$.*

Proof. For $u = u_0 + \sum \partial_{x_j} u_j$, integrating by parts, implies that

$$\left\|G^\psi(\cdot,\xi) * u\right\|_{L^\infty} \leq \left\|G^\psi(\cdot,\xi)\right\|_{L^1}\left\|u_0\right\|_{L^\infty} + \sum_{j=1}^d \left\|\partial_{x_j} G^\psi(\cdot,\xi)\right\|_{L^1}\left\|u_j\right\|_{L^\infty}.$$

Since $\left\|\partial_{x_j} G^\psi(\cdot,\xi)\right\|_{L^1} \leq C(1+|\xi|)$, this implies that

$$\left\|G^\psi(\cdot,\xi) * u\right\|_{L^\infty} \leq C(1+|\xi|)\left\|u\right\|_{W^{-1,\infty}}.$$

Applied to $u = a(\cdot,\xi)$, this implies that

$$\left\|\sigma_a^\psi(\,\cdot\,,\xi)\right\|_{L^\infty} \leq C(1+|\xi|)\left\|a(\,\cdot\,,\xi)\right\|_{W^{-1,\infty}}.$$

The estimates of the ξ derivatives are similar and the proposition follows □

5.1.3 Operators

Definition 5.1.14. *Let ψ be an admissible cut-off function. For $a \in \Gamma_0^m$ the paradifferential operator T_a^ψ is defined by*

$$T_a^\psi u(x) := \frac{1}{(2\pi)^d} \int e^{i\xi\cdot x}\, \sigma_a^\psi(x,\xi)\, \widehat{u}(\xi)\, d\xi\,. \tag{5.1.24}$$

Thus $T_a^\psi = \sigma_a^\psi(x, D_x)$. Since $\sigma_a^\psi \in \Sigma_0^m$, we can apply Theorem 4.3.5:

Theorem 5.1.15 (Action)**.** *Suppose that ψ is an admissible cut-off.*

i) When $a(\xi)$ is a symbol independent of x, the operator T_a^ψ is defined by the action of the Fourier multiplier a.

ii) For all $s \in \mathbb{R}$, $\mu \notin \mathbb{Z}$ with $\mu + m \notin \mathbb{Z}$ and all $a \in \Gamma_0^m$, T_a^ψ is a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and from $W^{m+\mu,\infty}(\mathbb{R}^d)$ to $W^{\mu,\infty}(\mathbb{R}^d)$.

To simplify the exposition we use the following terminology:

Definition 5.1.16. *An operator T is said of order $\leq m \in \mathbb{R}$ if it is bounded from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ and from $W^{m+\mu,\infty}(\mathbb{R}^d)$ to $W^{\mu,\infty}(\mathbb{R}^d)$ for all $\mu \notin \mathbb{Z}$ such that $\mu + m \notin \mathbb{Z}$.*

In particular, with this terminology, T_a^ψ is an operator of order $\leq m$ when $a \in \Gamma_0^m$. In addition, Theorem 4.3.5 also provides precise estimates: given s, m and ψ, there is a constant C such that for all $a \in \Gamma_0^m$ and $u \in H^{s+m}$

$$\left\| T_a^\psi u \right\|_{H^s} \leq C M_0^m(a; [\tfrac{d}{2}] + 1)\, \left\| u \right\|_{H^{s+m}}. \tag{5.1.25}$$

Similarly, Theorem 4.3.5 applied to $\sigma_a^{\psi_1} - \sigma_a^{\psi_2}$, implies the following result.

Proposition 5.1.17. *If ψ_1 and ψ_2 are two admissible cut-off, then for all $a \in \Gamma_r^m$, $s \in \mathbb{R}$ and $\mu \notin \mathbb{Z}$ with $\mu + m \notin \mathbb{Z}$, $T_a^{\psi_1} - T_a^{\psi_2}$ is of order $m - r$.*

In particular, for all s there is a constant C such that for all $a \in \Gamma_r^m$ and $u \in H^{s+m}$

$$\left\| (T_a^{\psi_1} - T_a^{\psi_2}) u \right\|_{H^s} \leq C M_r^m(a; [\tfrac{d}{2}] + 1)\, \left\| u \right\|_{H^{s+m-r}}. \tag{5.1.26}$$

Remark 5.1.18. This proposition implies that the choice of ψ is essentially irrelevant in our analysis, as long as one can neglect r-smoothing operators. (see [Bon]). To simplify notations, we make a definite choice of ψ, for instance $\psi = \psi_N$ with $N = 3$ as in (5.1.9) and we use the notation T_a for T_a^ψ.

5.2 Paraproducts

5.2.1 Definition

A function $a(x) \in L^\infty$ can be seen as a symbol in Γ^0_0, independent of (ξ, γ). With ψ given by (5.1.9) with $N = 3$, this leads to define

$$T_a\, u \,= S_{-3}a\, S_0 u + \sum_{k=1}^{\infty} S_{k-3}a\, \Delta_k\, u\,. \tag{5.2.1}$$

with $S_k = \chi_k(D_x)$ and $\Delta_k = S_k - S_{k-1}$.

Proposition 5.2.1. *For all $a \in L^\infty$, T_a is of order ≤ 0 and for all s there is a constant C such that*

$$\|T_a u\|_{H^s} \le C\|a\|_{L^\infty}\ \|u\|_{H^s}. \tag{5.2.2}$$

5.2.2 Products

For the *para*-product $T_a u$, only the L^∞ norm of a appears. This is in sharp contrast with the actual *product* au, which we now analyze. For functions in $\mathscr{S}$, there holds

$$au = T_a u + R_u a, \qquad \text{with} \quad R_u a := \sum_{k=-2}^{\infty} \Delta_k a\, S_{k+2} u. \tag{5.2.3}$$

Proposition 5.2.2. *The bilinear operator $R_u a$ extends to $a \in \mathscr{S}'$ such that $\nabla a \in H^{s'-1}$ with $s' > 0$ and $u \in L^\infty$, in which case $R_u a \in H^{s'}$, and there is a constant C such that*

$$\|R_u a\|_{H^{s'}} \le C\|\nabla a\|_{H^{s'-1}}\ \|u\|_{L^\infty}. \tag{5.2.4}$$

Proof. The sum $R_u a$ is quite similar to a paraproduct $T_u a$, except for two things:

- only terms in $\Delta_k a$ appear,
- the term $\Delta_k a\, S_{k+2} u$ has is spectrum in a ball $\{|\xi| \le 2^{k+4}\}$.

Therefore, using that

$$\begin{aligned} &\|S_{k+2} u\|_{L^\infty} \le C\|u\|_{L^\infty}, \\ &\|\Delta_k a\|_{L^2} \le C\varepsilon_k 2^{-ks'}, \qquad \sum_{k=-2}^{+\infty} \varepsilon_k^2 \le \|\nabla a\|^2_{H^{s'-1}} \end{aligned} \tag{5.2.5}$$

and applying Proposition 4.1.12 for positive indices s' implies the estimate (5.2.4). □

Together with the Sobolev embedding $H^s \subset L^\infty$ when $s > \frac{d}{2}$, Propositions 5.2.1 and 5.2.2 imply the following result:

Corollary 5.2.3. *i) For $s > \frac{d}{2}$, $H^s(\mathbb{R}^d)$ is a Banach algebra for the multiplication of functions.*

ii) If $s' > s > \frac{d}{2}$, then for $u \in H^s(\mathbb{R}^d)$ and $a \in H^{s'}(\mathbb{R}^d)$, the difference $au - T_a u$ belongs to $H^{s'}(\mathbb{R}^d)$.

More generally, the Littlewood-Paley decomposition is a powerful tool (though not universal) to analyze the products au when a and u are in Sobolev spaces (or Besov spaces). The results above are just examples of what can be done, but they are sufficient for our purposes in the next chapters.

5.2.3 Para-linearization 1

A key observation of J.M.Bony ([Bon], see also [Mey]) is that para-differential operators naturally arise when one perform a spectral analysis of nonlinear functionals. The main objective of this section is to prove the following theorem.

Theorem 5.2.4. *Let F be a C^∞ function on $\mathbb{R}$ such that $F(0) = 0$. If $u \in H^s(\mathbb{R}^d)$, with $\rho := s - \frac{d}{2} > 0$, then*

$$F(u) - T_{F'(u)}u \in H^{s+\rho}(\mathbb{R}^d)\,. \tag{5.2.6}$$

There is an analogous result in the scale of spaces $W^{\mu,\infty}$:

Theorem 5.2.5. *Let F be a C^∞ function on $\mathbb{R}$. If $u \in W^{\mu,\infty}(\mathbb{R}^d)$, with $\mu > 0$ $\mu \notin \frac{1}{2}\mathbb{N}$, then*

$$F(u) - T_{F'(u)}u \in W^{2\mu,\infty}(\mathbb{R}^d)\,. \tag{5.2.7}$$

In a preliminary step we prove the weaker result:

Theorem 5.2.6. *Let F be a C^∞ function on $\mathbb{R}$ such that $F(0) = 0$. If $u \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $s \geq 0$, then $F(u) \in H^s(\mathbb{R}^d)$ and*

$$\|F(u)\|_{H^s} \leq C_s\big(\|u\|_{L^\infty}\big)\|u\|_{H^s}. \tag{5.2.8}$$

Proof. When $s = 0$ the result is easy: since $F(0) = 0$, there is a smooth function G such that $F(u) = uG(u)$. In this product, $u \in L^2$ and $G(u) \in L^\infty$ since $u \in L^\infty$.

When $s > 0$ we use the notations $u_k = S_k u$ and $v_k = \Delta_k$ so that $u = \sum u_{k+1} - u_k$.

There are constants C_α independent of u and k such that:

$$\|\partial_x^\alpha v_k\|_{L^2} \leq C_\alpha 2^{(|\alpha|-s)k} \varepsilon_k \tag{5.2.9}$$

where $\sum \varepsilon_k^2 = \|u\|_{H^s}^2$. Moreover,

$$\|\partial_x^\alpha u_k\|_{L^\infty} \leq C_\alpha 2^{|\alpha|k} \|u\|_{L^\infty} . \tag{5.2.10}$$

with other constants C_α.

Because $u_k \to u$ in L^2 and the u_k and u are uniformly bounded in L^∞, there holds $F(u_k) \to F(u)$ in L^2 and thus

$$F(u) = F(u_0) + \sum_{k=0}^{\infty} \big(F(u_{k+1}) - F(u_k)\big) = F(u_0) + \sum_{k=0}^{\infty} m_k v_k \tag{5.2.11}$$

with

$$m_k = \int_0^1 F'(u_k + t v_k) dt. \tag{5.2.12}$$

By (5.2.9) and the chain rule, there are constants $C_\alpha = C_\alpha(\|u\|_{L^\infty})$, depending only on α and the L^∞ norm of u such that

$$\|\partial_x^\alpha F'(u_k + t v_k)\|_{L^\infty} \leq C_\alpha 2^{|\alpha|k} .$$

uniformly in t. Integrating, this shows that

$$\|\partial_x^\alpha m_k\|_{L^\infty} \leq C_\alpha 2^{|\alpha|k} \tag{5.2.13}$$

Therefore, with (5.2.9) we obtain that

$$\|\partial_x^\alpha (m_k v_k)\|_{L^2} \leq C_\alpha 2^{(|\alpha|-s)k} \varepsilon_k \tag{5.2.14}$$

with new constants $C_\alpha = C_\alpha(\|u\|_{L^\infty})$, and the Theorem follows, using Proposition 4.1.13. □

To prove Theorem 5.2.4, we first note that there is no restriction in assuming that $F'(0) = 0$, since this amounts to add or subtract a fixed linear term au to $F(u)$. Next, because $\rho > 0$, the Sobolev injection theorem implies that $u \in L^\infty(\mathbb{R}^d)$. By definition,

$$T_{F'(u)} u = S_{-3} g \; u_0 + \sum_{k=0}^{\infty} S_{k-2} g \; v_k \tag{5.2.15}$$

with $g = F'(u)$ (see (5.2.1)). We compare this expression with (5.2.11). The first terms $F(u_0)$ and $S_{-3}g\, u_0$ belong to H^∞. Thus it remains to prove that

$$\sum_{k=0}^{\infty} (m_k - S_{k-2}g)\, v_k \ \in \ H^{s+\rho}. \tag{5.2.16}$$

Using the L^2 estimates (5.2.9) for the derivatives of v_k, the conclusion follows from Proposition 4.1.13 and the following L^∞ estimates for the derivatives of $(m_k - S_{k-2}g)$:

$$\big\| \partial_x^\alpha (m_k - S_{k-2}g) \big\|_{L^\infty} \le C_\alpha 2^{(|\alpha| - \rho)k}. \tag{5.2.17}$$

To prove these estimates, we split $m_k - S_{k-2}g$ into two terms $\big(m_k - F'(u_{k-2})\big) + \big(F'(u_{k-2}) - S_{k-2}g\big)$ and we study them separately.

Lemma 5.2.7. *There holds, with constants independent of k:*

$$\big\| \partial_x^\alpha \big(m_k - F'(u_{k-2})\big) \big\|_{L^\infty} \le C_\alpha 2^{(|\alpha| - \rho)k}. \tag{5.2.18}$$

$$\big\| \partial_x^\alpha \big(F'(S_k u) - S_k F'(u)\big) \big\|_{L^\infty} \le C_\alpha 2^{(|\alpha| - \rho)k}. \tag{5.2.19}$$

Proof. **a)** Taylor's theorem implies that

$$F'(u_k + t v_k) - F'(u_{k-2}) = \mu_k w_k$$

with

$$w_k = (v_{k_2} + v_{k-1} + t v_k) \quad \text{and} \quad \mu_k = \int_0^1 F''(u_{k-2} + \tau w_k) d\tau.$$

The μ_k satisfy estimates similar to (5.2.13)

$$\| \partial_x^\alpha \mu_k \|_{L^\infty} \le C_\alpha 2^{|\alpha| k}$$

while the w_k satisfy

$$\| \partial_x^\alpha w_k \|_{L^\infty} \le C 2^{\frac{d}{2}} \| \partial_x^\alpha w_k \|_{L^2} \le C_\alpha 2^{|\alpha| - \rho k} \varepsilon_k \le \tilde{C}_\alpha 2^{(|\alpha| - \rho)k}. \tag{5.2.20}$$

with $\sum \varepsilon_k^2 = \|u\|_{H^s}^2$. Thus

$$\| \partial_x^\alpha (\mu_k w_k) \|_{L^\infty} \le C_\alpha 2^{(|\alpha| - \rho)k}.$$

These estimates are uniform in $t \in [0,1]$. Since

$$m_k = \int_0^1 \mu_k w_k dt$$

this implies (5.2.18).

b) To prove (5.2.19) we split the term to estimate into two pieces:

$$G(u_k) - S_k G(u_k) - S_k\big(G(u) - G(u_k)\big) \tag{5.2.21}$$

with $G = F'$. There holds

$$\|\partial_x^\alpha S_k\big(G(u) - G(u_k)\big)\|_{L^\infty} \lesssim 2^{(|\alpha| + \frac{d}{2})k} \|S_k\big(G(u) - G(u_k)\big)\|_{L^2}$$

and

$$\|S_k\big(G(u) - G(u_k)\big)\|_{L^2} \lesssim \|G(u) - G(u_k)\|_{L^2} \lesssim \|u - u_k\|_{L^2} \lesssim 2^{-ks}.$$

This implies that the second term in (5.2.21) satisfies (5.2.19).

Next we note that $u_k \in H^{s+N}$ for all N and that

$$\|u_k\|_{H^{s+N}} \le C_N 2^{kN}, \qquad \|u_k\|_{L^\infty} \le C \tag{5.2.22}$$

with C and C_N independent of k. Therefore, by Theorem 5.2.6, $G(u_k) \in H^{s+N}$ and

$$\|G(u_k)\|_{H^{s+N}} \le C_N 2^{kN}.$$

We use the following estimate, valid for $|\alpha| < \sigma - \frac{d}{2}$ and $a \in H^\sigma$:

$$\|\partial_x^\alpha (a - S_k a)\|_{L^\infty} \le C 2^{k(|\alpha| - \sigma + \frac{d}{2})} \|a\|_{H^\sigma}. \tag{5.2.23}$$

Indeed, $a - S_k a = \sum_{j \ge k} \Delta_j a$ and

$$\|\partial_x^\alpha \Delta_j a\|_{L^\infty} \le C 2^{j(|\alpha| - \sigma + \frac{d}{2})} \|a\|_{H^\sigma}$$

so that the series is converges if $|\alpha| < \sigma - \frac{d}{2}$.

Applied to $a = G(u_k)$ and $\sigma = s + N$ with N sufficiently large, this estimate yields

$$\|\partial_x^\alpha \big(G(u_k) - S_k G(u_k)\big)\|_{L^\infty} \lesssim 2^{k(|\alpha| - s - N + \frac{d}{2})} \|G(u_k)\big)\|_{H^{s+N}}.$$

Together with the estimate of the H^{s+N} norm of $G(u_k)$, this implies that the first term in (5.2.21) also satisfies (5.2.19).

This finishes the proof of the Lemma and therefore of Theorem 5.2.4 □

The proof of Therorem 5.2.5 is quite similar and omitted.

5.2.4 Para-linearization 2

We give here another useful result, which allows to replace a product au by a para-product $T_a u$, to the price of a smoother term. This was already done in Corollary 5.2.3 for $a \in H^{s'}$. Here we consider the case where a belongs to a space $W^{r,\infty}$.

Theorem 5.2.8. *Let r be a positive integer. There is a constant C such that for $a \in W^{r,\infty}$, the mapping $u \mapsto au - T_a u$ extends from L^2 to H^r and*

$$\big\| au - T_a u \big\|_{H^r} \;\le\; C \|a\|_{W^{r,\infty}} \big\| u \big\|_{L^2}. \tag{5.2.24}$$

Theorem 5.2.9. *Let r be a positive integer. There is C such that for $a \in W^{r,\infty}$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| \le r$, the mapping $u \mapsto a\partial_x^\alpha u - T_a \partial_x^\alpha u$ extends from L^2 to L^2 and*

$$\big\| a\partial_x^\alpha u - T_a \partial_x^\alpha u \big\|_{L^2} \;\le\; C \big\| a \big\|_{W^{r,\infty}} \big\| u \big\|_{L^2}\,. \tag{5.2.25}$$

Proof of Theorem 5.2.8 . Start from the identity

$$au - T_a u \;=\; \sum_{k=-2}^{\infty} \Delta_k a\, S_{k+2} u = f + g\,.$$

with

$$f = \sum_{k=-2}^{\infty} f_k\,, \qquad f_k := \sum_{|j-k|\le 2} \Delta_k a\, \Delta_j u\,, \tag{5.2.26}$$

$$g = \sum_{k=-2}^{\infty} g_k, \qquad g_k := \Delta_k a\, S_{k-3} u \tag{5.2.27}$$

We prove that f and g belong to H^r separately.

We first consider f. Propositions 4.1.11, 4.1.16 and 4.1.19 imply that

$$\|f_k\|_{L^2} \le \; C\, 2^{-k} \|a\|_{W^{1,\infty}}\, \rho_k \qquad \rho_k := \sum_{|j-k|\le 2} \|\Delta_j u\|_{L^2}\,.$$

Moreover,

$$\sum_k \rho_k^2 \le C \sum_j \|\Delta_j u\|_{L^2}^2 \;\le\; C\|u\|_{L^2}^2\,.$$

The spectrum of $\Delta_k a$ is contained in the ball $2^{k-1} \le |\xi| \le 2^{k+1}$ and the spectrum of $\Delta_j u$ is containded in $2^{j-1} \le |\xi| \le 2^{j+1}$. Therefore, the spectrum of f_k in contained in the ball $|\xi| \le 2^{k+4}$. Hence, Proposition 4.1.12 implies that $f \in H^r$ and

$$\big\| f \big\|_{H^r} \le C \big\| a \big\|_{W^{r,\infty}} \big\| u \big\|_{L^2} .$$

It remains to prove a similar estimate for g: we prove that for $|\alpha| \le r$, there holds

(5.2.28) $$\big\| \partial_x^\alpha g \big\|_{L^2} \le C \big\| a \big\|_{W^{r,\infty}} \big\| u \big\|_{L^2} .$$

The derivative $\partial_x^\alpha g$ is a linear combination of terms

(5.2.29) $$g_{\alpha,\beta} = \sum_k \Delta_k \partial_x^{\alpha-\beta} a \ S_{k-3}^\gamma \partial_x^\beta u ,$$

1) Consider the case $|\beta| > 0$. By Proposition 4.1.16 and Corollary 4.1.7, there holds

$$\big\| \Delta_k \partial_x^{\alpha-\beta} a \big\|_{L^\infty} \le C 2^{-k(r-|\alpha|+|\beta|)} \big\| a \big\|_{W^{r,\infty}} .$$

Moreover,

$$\big\| S_{k-3} \partial_x^\beta u \|_{L^2} \le \big\| S_{-3} \partial_x^\beta u \|_{L^2} + \sum_{j \le k-3} \big\| \Delta_j \partial_x^\beta u \|_{L^2} \le C \sum_{j \le k-3} 2^{j|\beta|} \varepsilon_j$$

with $\sum \varepsilon_j^2 \le \|u\|_{L^2}^2$. Since $|\beta| > 0$, note that that

$$\sum_{j \le k-3} 2^{j|\beta|} \varepsilon_j = 2^{k|\beta|} \rho_k \quad \text{with} \quad \sum \rho_k^2 \le \|u\|_{L^2}^2 .$$

Thus $w_k = \Delta_k \partial_x^{\alpha-\beta} a \ S_{k-3} \partial_x^\beta u$ satisfies

$$\big\| w_k \big\|_{L^2} \le C 2^{k(r-|\alpha|)} \rho_k \big\| a \big\|_{W^{r,\infty}} .$$

Moreover, its spectrum of is contained in $\{2^{k-2} \le (1+|\xi|^2)^{1/2} \le 2^{k+2}\}$. Hence, Proposition 4.1.12 implies that $g_{\alpha,\beta} = \sum w_k \in L^2$ and

(5.2.30) $$\big\| g_{\alpha,\beta} \big\|_{L^2} \le C \big\| a \big\|_{W^{r,\infty}} \big\| u \big\|_{L^2} .$$

2) Finally, consider the case $\beta = 0$. Let $b = \partial_x^\alpha a \in L^\infty$. The spectrum of $\Delta_k a \, S_{k-3} u$ is contained in $\{2^{k-2} \le (1+|\xi|^2)^{1/2} \le 2^{k+2}\}$. Therefore

$$\big\| g_{\alpha,0} \big\|_{L^2}^2 \le C \sum_k \big\| \Delta_k b \, S_{k-3} u \big\|_{L^2}^2$$

To prove that $g_{\alpha,0}$ satisfies the estimate (5.2.30), it is therefore sufficient to prove that

$$\sum_k \big\| \Delta_k b \, S_{k-3} u \big\|_{L^2}^2 \le C \|a\|_{W^{1,\infty}}^2 \|u\|_{L^2}^2 .$$

This estimate is a consequence of the next two results which therefore complete the proof of Theorem 5.2.8. □

Theorem 5.2.10. *There is a constant C such that for all $b \in L^\infty$ and all sequence v_k in L^2 one has*

$$\int \sum_{k\geq 1} |\Delta_k b(x)|^2 \, |v_k(x)|^2 dx \ \leq \ C\|b\|^2_{L^\infty} \, \|v_*\|^2_{L^2} \tag{5.2.31}$$

with

$$v_*(x) := \sup_{k\geq 1} \sup_{|y-x|\leq 2^{-k}} |v_k(y)| \, . \tag{5.2.32}$$

Lemma 5.2.11. *Consider $u \in L^2$, $v_k = S_k u$ and define v_* by (5.2.32). Then there is a constant C such that*

$$v_*(x) \leq Cu^*(x) \tag{5.2.33}$$

where u^* is the maximal function

$$u^*(x) := \sup_R \frac{1}{R^n} \int_{|y-x|\leq R} |u(y)| dy$$

In particular, $v_* \in L^2$ and there is a constant C such that $\|v_*\|_{L^2} \leq C\|u\|_{L^2}$.

In [CM] it is proved that when $b \in BMO$, $\sum_k |\Delta_k b(x)|^2 \otimes \delta_{t=2^{-k}}$ is a Carleson measure which immediately implies (5.2.31) The fact that the maximal function u^* belongs to L^2 when $u \in L^2$ is also a well known result from Harmonic Analysis (see e.g. [CM, Ste]). For the sake of completeness, we include a short proof of the estimate(5.2.31) in the easier case when $b \in L^\infty$.

Proof of Theorem (5.2.10)*.*

a) We show that for all open set $\Omega \subset \mathbb{R}^d$:

$$\sum_{k>0} \|\Delta_k b\|^2_{L^2(\Omega_k)} \leq C \ \text{meas}(\Omega) \, \|b\|^2_{L^\infty} \, , \tag{5.2.34}$$

where Ω_k denotes the set of points $x \in \Omega$ such that the ball $B(x, 2^{-k}) := \{y \in \mathbb{R}^d \, : \, |x-y| < 2^{-k}\}$ is contained in Ω.

Write $b = b' + b''$ with $b' = b 1_\Omega$. Denote by $I(b)$ the left hand side of (5.2.34). Then $I(b) \leq 2I(b') + 2I(b'')$. Therefore, it is sufficient to prove the inequality separately for b' and b''. One has

$$\sum_k \|\Delta_k b'\|^2_{L^2(\Omega_k)} \leq \sum_k \|\Delta_k b'\|^2_{L^2(\mathbb{R}^d)} \leq \|b'\|^2_{L^2} \leq \|b\|^2_{L^\infty} \, \text{meas}(\Omega) \, .$$

Thus, it remains to prove (5.2.34) for b''.

The kernel of Δ_k is $G_k(x) = 2^{kd}G_0(2^k x)$ where G_0 belongs to the Schwartz'class $\mathscr{S}(\mathbb{R}^d)$. Thus

$$\Delta_k b''(x) = \int 2^{kd} G_0(2^k(x-y)) b''(y)\, dy\,.$$

On the support of b'', $y \notin \Omega$ and for $x \in \Omega_l$, the distance $|x-y|$ is larger than 2^{-l}. Thus, for $x \in \Omega_l$

$$|\Delta_k b''(x)| \leq \|b''\|_{L^\infty} \int_{\{|y|\geq 2^{-l}\}} 2^{kd}\, |G_0(2^k y)|\, dy = \|b''\|_{L^\infty}\, g^*_{k-l}$$

with

$$g^*_l = \int_{\{|y|\geq 2^l\}} |G_0(y)|\, dy.$$

Let $\Omega'_0 := \Omega_0$ and for $l > 0$, let $\Omega'_l = \Omega_l \setminus \Omega_{l-1}$. Then the pointwise estimate above implies that

$$\|\Delta_k b''\|^2_{L^2(\Omega'_l)} \leq \|b\|^2_{L^\infty} \operatorname{meas}(\Omega'_l)\, \big(g^*_{k-l}\big)^2. \tag{5.2.35}$$

Since $\Omega_k = \bigcup_{l\leq k} \Omega'_l$,

$$\sum_{k\geq 1} \|\Delta_k b''\|^2_{L^2(\Omega_k)} = \sum_{k\geq 1} \sum_{l=0}^{k} \|\Delta_k b''\|^2_{L^2(\Omega'_l)}\,.$$

With (2.4.26), this shows that

$$\sum_{k>0} \|\Delta_k b''\|^2_{L^2(\Omega_k)} \leq \sum_{l\geq 0} \sum_{k\geq l} \|b\|^2_{L^\infty} \big(g^*_{k-l}\big)^2 \operatorname{meas}(\Omega'_l)\,.$$

Since $G_0 \in \mathscr{S}$, the sequence g^*_k is rapidly decreasing and thus in $\ell^2(\mathbb{N})$. Therefore,

$$\sum_{k>0} \|\Delta_k b''\|^2_{L^2(\Omega_k)} \leq C\, \|b\|^2_{L^\infty} \sum_{l\geq 0} \operatorname{meas}(\Omega'_l) = C\, \|b\|^2_{L^\infty} \operatorname{meas}(\Omega)\,.$$

showing that b'' also satisfies (5.2.34).

b) Let $b_k = \Delta_k b$. Then

$$\|b_k v_k\|^2_{L^2} = 2 \int_0^\infty \lambda \|b_k\|^2_{L^2(U_k(\lambda))}\, d\lambda\,, \quad \text{where} \quad U_k(\lambda) = \{|v_k| > \lambda\}$$

For $\lambda > 0$, let $\Omega(\lambda) = \{|v_*| > \lambda\}$. This is the set of points x such that there are $k > 0$ and y such that $|x - y| < 2^{-k}$ and $|v_k(y)| > \lambda$. Thus $\Omega(\lambda)$ is open and if $|v_k(y)| > \lambda$, the ball $B(y, 2^{-k})$ is contained in $\Omega(\lambda)$. This shows that for all k, $U_k(\lambda) \subset \Omega_k(\lambda)$, where the Ω_k's are defined as in (5.2.34). Thus

$$\sum_{k>0} \|b_k\|^2_{L^2(U_k(\lambda))} \leq \sum_{k>0} \|b_k\|^2_{L^2(\Omega_k(\lambda))} \leq C \|b\|^2_{L^\infty} \operatorname{meas}(\Omega(\lambda))$$

and

$$\sum_{k>0} \|b_k v_k\|^2_{L^2} \leq 2C \|b\|^2_{L^\infty} \int_0^\infty \lambda \operatorname{meas}(\Omega(\lambda))\, d\lambda = C \|b\|^2_{L^\infty} \|v_*\|^2_{L^2}$$

which is (5.2.31). □

Proof of Lemma 5.2.11. S_k^γ is the convolution operator with $\varphi\chi_k$, the inverse Fourier transform of $\chi(2^{-k}\xi)$. Thus there is C such that

$$|\varphi_k(x)| \; \leq \; C\, 2^{dk}(1 + 2^k|x|)^{-d-1}\,.$$

Thus

$$|v_k(x - x')| \leq C2^{dk} \int (1 + 2^k|y - x'|)^{-d-1}\, |u(x - y)| dy\,.$$

Splitting the domain of integration into annuli $|y| \approx 2^{j-k}$, $j \geq 0$ implies that

$$\sup_{|x'| \leq 2^{-k}} |v_k(x - x')| \; \leq \; C'\, 2^{dk} \sum_{j \geq 0} 2^{-j(d+1)} 2^{d(j-k)} u^*(x)$$

and the lemma follows. □

Proof of Theorem 5.2.9.

Use the notation $E(a, u) = au - T_a u$. We show by induction on $|\alpha| \leq r$ that for

$$\big\|E(a, \partial_x^\alpha u)\big\|_{H^{r-|\alpha|}} \leq C\big\|a\big\|_{W^{r,\infty}} \; \big\|u\big\|_{L^2}. \tag{5.2.36}$$

For $r > 0$ and $\alpha = 0$ this is Theorem 5.2.8. When $r = \alpha = 0$, each term au and $T_a u$ belongs to L^2 when $a \in L^\infty$ and $u \in L^2$. Suppose that is proved for $|\alpha| \leq l < r$. The definition (5.2.1) of the para-product implies that

$$\begin{aligned} a\partial_{x_j}\partial_x^\alpha u - T_a\partial_{x_j}\partial_x^\alpha u =& \partial_{x_j}\big(a\partial_x^\alpha u - T_a\partial_x^\alpha u\big) \\ & - (\partial_{x_j} a)\partial_x^\alpha u + T_{\partial_{x_j} a}\partial_x^\alpha u \end{aligned}$$

that is

$$E(a, \partial_{x_j} \partial_x^\alpha u) = \partial_{x_j} E(a, \partial_x^\alpha u) - E(\partial_{x_j} a, \partial_x^\alpha u).$$

By the induction hypothesis, $E(a, \partial_x^\alpha u) \in H^{r-|\alpha|}$ and $\partial_{x_j} E(a, \partial_x^\alpha u) \in H^{r-|\alpha|-1}$. The induction hypothesis also implies that $E(\partial_{x_j} a, \partial_x^\alpha u) \in H^{r-1-|\alpha|}$. Thus (5.2.36) follows at the order $l + 1$, finishing the proof of the theorem. $\square$

Chapter 6

Symbolic calculus

The symbolic calculus is what makes the theory efficient and useful. The idea of symbolic calculus is to replace the calculus on operators (composition, adjoints, inverses...) by a calculus on the symbols. This was done in Lemmas 4.2.2 and 4.2.3 for operators with symbols in the Schwartz class, but the exact formulas (4.2.6) or (4.2.10) involve integrals and Fourier transforms and are not easily usable. On the other hand, Proposition 3.2.3 answers exactly the questions for Fourier multipliers : the calculus on operator is isomorphic to an algebraic calculus on symbols. This is what one wants to extend. The classical theory of pseudo-differential operators shows that there are simple and usable, but approximate, formulas, if one allows for *error terms* which are of order strictly less than the order of the main term. This idea is already present in the definition of para-differential operators : the operator T_a^ψ depends on the choice of ψ, but Proposition 5.1.17 implies that for $a \in \Gamma_r^m$, changing the admissible function ψ would modify T_a, which is of order m, by an operators of order $m-r$. The main purpose of this chapter is to provide a symbolic calculus for symbols of limited smoothness r, modulo errors terms which are r-smoother than the main term: composition (Theorem 6.1.1), adjoints (Theorem 6.2.1). Next we provide applications to elliptic estimates and Gårding's inequality (Theorem 6.3.4) which will be also very important in the next part.

6.1 Composition

6.1.1 Statement of the result

Theorem 6.1.1. *Consider $a \in \Gamma_r^m$ and $b \in \Gamma_r^{m'}$, where $r > 0$.*

i) The symbol

$$a\sharp b := \sum_{|\alpha|<r} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi) \tag{6.1.1}$$

is well defined in $\sum_{j<r}\Gamma_{r-j}^{m+m'-j}$.

ii) $T_a \circ T_b - T_{a\sharp b}$ *is of order* $\leq m+m'-r$.

This extends to matrix valued symbols and operators.

In (6.1.1), $D_x = \frac{1}{i}\partial_x$

Because $W^{r,\infty}(\mathbb{R}^d)$ is a Banach algebra, we first note that

$$a\in\Gamma_r^m,\ b\in\Gamma_r^{m'} \quad\Rightarrow\quad ab\in\Gamma_r^{m+m'}. \tag{6.1.2}$$

For $a\in\Gamma_r^m$, $b\in\Gamma_r^{m'}$ and $|\alpha|<r$:

$$\partial_\xi^\alpha a\in\Gamma_r^{m-|\alpha|},\quad D_x^\alpha b\in\Gamma_{r-|\alpha|}^{m'}, \tag{6.1.3}$$

so that

$$\partial_\xi^\alpha a\, D_x^\alpha b\ \in\Gamma_{r-|\alpha|}^{m+m'-|\alpha|}. \tag{6.1.4}$$

Therefore

$$a\sharp b=\sum_{j<r}c_j,\qquad c_j=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\in\Gamma_{r-j}^{m+m'-j}. \tag{6.1.5}$$

This proves the first part of the theorem.

We will check that the result in *ii)* does not depend on the choice of the admissible cut-off function ψ used to define para-differential quantization T. In a first step we study the composition of operators with symbols which satisfy the spectral condition.

6.1.2 Proof of the main theorem

Lemma 6.1.2. *Suppose that* $p\in\Sigma_0^m$ *and* $q\in\Sigma_0^{m'}$. *Let* θ *be an admissible cut-off function such that* $\theta=1$ *on a neighborhood of the support of* $\mathscr{F}_x q$. *Let*

$$\sigma(x,\xi)\ =\int H(x,y,\xi)\,q(y,\xi)\,dy \tag{6.1.6}$$

with

$$H(x,y,\xi)\ :=\ \frac{1}{(2\pi)^d}\int e^{i(x-y)\eta}p(x,\xi+\eta)\theta(\eta,\xi)\,d\eta\,.$$

belongs to $S^{m+m'}_{1,1}$ *and*

$$p(x, D_x) \circ q(x, D_x) = \sigma(x, D_x) \tag{6.1.7}$$

Moreover, if p *and* q *satisfy the spectral condition* (4.2.16) *with parameters* ε *and* ε' *and if* $\varepsilon + \varepsilon' + \varepsilon\varepsilon' < 1$*, then* σ *satisfies the spectral property and* $\sigma \in \Sigma^{m+m'}_0$.

Proof. **a)** Using the approximation Lemma 4.2.11 it is sufficient to prove (6.1.6) when p and q belong to $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$. In this case, Lemma 4.2.2 implies that $p(x, D_x) \circ q(x, D_x) = \sigma(x, D_x)$ where the symbol σ is given by (4.2.7), that is

$$\begin{aligned} \sigma(x, \xi) &= e^{-ix\xi}\big(p(x, D_x)\rho_\xi\big)(x)\,, \\ &= \frac{1}{(2\pi)^d} \int e^{ix(\eta-\xi)} p(x, \eta)\hat{\rho}_\xi(\eta)\, d\eta\,, \end{aligned} \tag{6.1.8}$$

with $\rho_\xi(x) := e^{ix\xi} q(x, \xi)$. Since $\theta = 1$ on the support of $\mathscr{F}_x q$, there holds

$$\begin{aligned} \hat{\rho}_\xi(\eta) = \mathscr{F}_x q(\eta - \xi, \xi) &= \theta(\eta - \xi, \xi)\mathscr{F}_x q(\eta - \xi, \xi) \\ &= \int e^{i(\xi-\eta)y}\theta(\eta - \xi, \xi) q(y, \xi) dy. \end{aligned}$$

Substituting in (6.1.8) yields (6.1.6).

b) Because $|\eta| < \varepsilon''(1 + |\xi|)$ for some $\varepsilon'' < 1$ on the support of θ, the function $r(x, \xi, \eta) = p(x, \xi + \eta)\theta(\eta, \xi)$ satisfies

$$\big|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^{\beta'} r(x, \xi, \eta)\big| \le C_{\beta,\beta'}(1 + |\xi|)^{m+|\alpha|-|\beta|-|\beta'|}. \tag{6.1.9}$$

Thus, by Lemma 4.3.2, H satisfies estimates of the form

$$\big\|\partial_x^\alpha \partial_\xi^\beta H(x, \cdot\,, \xi)\big\|_{L^1(\mathbb{R}^d)} \le C_{\alpha,\beta}(1 + |\xi|)^{m+|\alpha|-|\beta|}\,. \tag{6.1.10}$$

Together with the estimates

$$\big\|\partial_\xi^\beta q(\cdot\,, \xi)\big\|_{L^\infty(\mathbb{R}^d)} \le C_\beta(1 + |\xi|)^{m'|-|\beta|},$$

this implies that $\sigma \in S^{m+m'}_{1,1}$.

c) The spectrum of $q(\cdot, \xi)$ is contained in $\{|\eta| \le \varepsilon'(1 + |\xi|)\}$, hence the spectrum of ρ_ξ which is translated from the previous one by by ξ is contained in $K = \{\eta : |\xi - \eta| \le \varepsilon_2(1 + |\xi|)\}$. Hence Lemma 4.2.12 implies that the spectrum of $p(x, D_x)\rho_\xi$ is contained in

$$\begin{aligned}\big\{\eta + \eta' \ : \ |\xi - \eta| \le \varepsilon'(1 + |\xi|),\ |\eta'| \le \varepsilon(1 + |\eta|)\big\} \\ \subset \big\{\zeta \ : \ |\xi - \zeta| \le (\varepsilon' + \varepsilon(1 + \varepsilon'))(1 + |\xi|)\big\}.\end{aligned}$$

Thus the spectrum of $\sigma(\,\cdot\,, \xi)$ is contained in $\{\eta : |\eta| \le \delta(1 + |\xi|)\}$ with $\delta = \varepsilon + \varepsilon' + \varepsilon\varepsilon'$ and therefore σ satisfies the spectral property (4.2.16) if $\delta < 1$.

□

Proof of Theorem 6.1.1.

a) We first check that the result does not depend on the choice of the admissible cut-off function ψ. Indeed, by Proposition 5.1.17 and Theorem 5.1.15,

$$T_a^{\psi_1} \circ T_b^{\psi_1} - T_a^{\psi_2} \circ T_b^{\psi_2} = \big(T_a^{\psi_1} - T_a^{\psi_2}\big) \circ T_b^{\psi_1} + T_a^{\psi_2} \circ \big(T_b^{\psi_1} - T_b^{\psi_2}\big) \tag{6.1.11}$$

is of order $\le m + m' - r$ and

$$T_{a\sharp b}^{\psi_1} - T_{a\sharp b}^{\psi_2} = \sum_{j<r} T_{c_j}^{\psi_1} - T_{c_j}^{\psi_2} \tag{6.1.12}$$

where the c_j are defined in (6.1.5). Since $c_j \in \Gamma_{r-j}^{m+m'-j}$ the difference $T_{c_j}^{\psi_1} - T_{c_j}^{\psi_2}$ is of order $\le m + m' - j - (r - j) = m + m' - r$.

Therefore, changing ψ if necessary, we now assume that the quantization T is associated to an admissible function ψ whose parameter ε_2 in (5.1.6) is smaller that $\frac{1}{4}$.

b) Let θ be another admissible cut-off function such that $\theta = 1$ on a neighborhood of the support of ψ. Let σ_a and σ_b denote the symbols associated to a and b. Then, $T_a^\psi = \sigma_a(x, D_x)$, $T_b^\psi = \sigma_b(x, D_x)$ and by the previous lemma, $T_a^\psi \circ T_b^\psi = \sigma(x, D_x)$ with

$$\sigma(x, \xi) \ = \ \int H(x, y, \xi)\, \sigma_b(y, \xi)\, dy$$

and

$$H(x, y, \xi,) \ := \ \frac{1}{(2\pi)^d} \int e^{i(x-y)\eta} \sigma_a(x, \xi + \eta)\theta(\eta, \xi)\, d\eta\,.$$

By Taylor's formula:

$$\sigma_a(x, \xi + \eta) = \sum_{|\alpha|<r} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_a(x, \xi)\eta^\alpha + \sum_{|\alpha|=\overline{r}} \rho_\alpha(x, \xi, \eta)\eta^\alpha, \tag{6.1.13}$$

where $\overline{r}$ is the smallest integer $\geq r$. Substituting, yields

$$\sigma = \sum_{|\alpha|<r} p_\alpha + \sum_{|\alpha|=\overline{r}} q_\alpha$$

with

$$\begin{aligned} p_\alpha(x,\xi) &= \frac{1}{(2\pi)^d \alpha!} \partial_\xi^\alpha \sigma_a(x,\xi) \int e^{i(x-y)\eta} \theta(\eta,\xi) \eta^\alpha \sigma_b(y,\xi)\, dy d\eta \\ &= \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_a(x,\xi) D_x^\alpha \sigma_b(x,\xi) \end{aligned}$$

and

$$\begin{aligned} q_\alpha(x,\xi) &= \frac{1}{(2\pi)^d} \int e^{i(x-y)\eta} \rho_\alpha(x,\xi,\eta) \theta(\eta,\xi) \eta^\alpha \sigma_b(y,\xi)\, dy d\eta \\ &= \int R_\alpha(x, x-y, \xi)\, (D_x^\alpha \sigma_b)(y,\xi)\, dy \end{aligned}$$

with

$$R_\alpha(x,y,\xi) \;=\; \frac{1}{(2\pi)^d} \int e^{iy\eta} \rho_\alpha(x,\xi,\eta) \theta(\eta,\xi)\, d\eta\,.$$

In the computation of p_α, we have used that $\theta(\,\cdot\,,\xi) = 1$ on the support of $\mathscr{F}_x \sigma_b(\,\cdot\,,\xi)$. Summing up, the Taylor expansion (6.1.13) implies the following decomposition of σ:

$$\sigma = \sigma_a \sharp \sigma_b + q, \tag{6.1.14}$$

with $q = \sum q_\alpha$.

c) Note that by Proposition 5.1.9

$$D_x^\alpha \sigma_b \in \Sigma_0^{m'+\overline{r}-r}, \tag{6.1.15}$$

so that

$$\left\| \partial_\xi^\beta \partial_x^\alpha \sigma_b(\,\cdot\,\xi) \right\|_{L^\infty(\mathbb{R}^d)} \;\leq\; C_{\alpha,\beta} (1+|\xi|)^{m'+\overline{r}-r-|\beta|}\,.$$

Moreover, in the Taylor expansion (6.1.13), the remainders ρ_α involve $\overline{r}$ ξ-derivatives of σ_a. Since θ is supported in $|\eta| \leq \frac{1}{4}(1+|\xi|)$, this implies that $r_\alpha(x,\xi,\eta) = \rho_\alpha(x,\xi,\eta)\theta(\eta,\xi)$ behaves like a symbol of degree $m - \overline{r}$ and satisfies estimates of the form

$$\left| \partial_\xi^\beta \partial_\eta^{\beta'} r_\alpha(x,\xi,\eta) \right| \leq C_{\alpha,\beta,\beta'} (1+|\xi|)^{m-\overline{r}-|\beta|-|\beta'|}.$$

Thus, by Lemma 4.3.2, R_α satisfies estimates

$$\left\| \partial_\xi^\beta R_\alpha(x,\,\cdot\,,\xi) \right\|_{L^1(\mathbb{R}^d)} \;\leq\; C_{\alpha,\beta} (1+|\xi|)^{m-\overline{r}-|\beta|}$$

and therefore

$$\big\|\partial_\xi^\beta q_\alpha(\,\cdot\,\xi)\big\|_{L^\infty(\mathbb{R}^d)} \le C_{\alpha,\beta}(1+|\xi|)^{m+m'-r-|\beta|}\,. \tag{6.1.16}$$

d) Next we use the following lemma.

Lemma 6.1.3. *If ψ satisfies* (5.1.6) *with $\varepsilon_1 < \frac{1}{2}$, then for $a \in \Gamma_r^m$ and $b \in \Gamma_r^{m'}$*

$$\sigma_a \sharp \sigma_b - \sigma_{a\sharp b} \in \Sigma_0^{m+m'-r}. \tag{6.1.17}$$

Combining with (6.1.14) and (6.1.16), it implies that $\sigma - \sigma_{a\sharp b}$ satisfies

$$\big\|\partial_\xi^\beta\big(\sigma - \sigma_{a\sharp b}\big)(\,\cdot\,\xi)\big\|_{L^\infty(\mathbb{R}^d)} \le C_\beta(1+|\xi|)^{m+m'-\bar{r}-|\beta|}\,.$$

Moreover, since we have chosen ψ such that $\varepsilon_1 < \frac{1}{4}$, Lemma 6.1.2 implies that σ satisfies the spectral property. Thus $\sigma - \sigma_{a\sharp b}$ also satisfies the spectral property and

$$\tilde{\sigma} := \sigma - \sigma_{a\sharp b} \in \Sigma_0^{m+m'-r}. \tag{6.1.18}$$

Therefore, $T_a \circ T_b - T_{a\sharp b} = \tilde{\sigma}(x, D_x)$ is of order $\le m + m' - r$. □

Proof of Lemma 6.1.3.

Let $a \in \Gamma_r^m$ and $b \in \Gamma_r^{m'}$. For $|\alpha| < r$, by Propositions 5.1.9 and (5.1.10)

$$\partial_\xi^\alpha \sigma_a - \sigma_{\partial_\xi^\alpha a} \in \Sigma_0^{m-|\alpha|-r}$$
$$D_x^\alpha \sigma_b = \sigma_{D_x^\alpha b} \in \Sigma_{r-|\alpha|}^{m'}.$$

Moreover, these symbols satisfy the spectral property with parameter $\varepsilon_1 < \frac{1}{2}$. Therefore, their products also satisfy the spectral property and

$$\partial_\xi^\alpha \sigma_a D_x^\alpha \sigma_b - \sigma_{\partial_\xi^\alpha a}\sigma_{D_x^\alpha b} \in \Sigma_0^{m+m'-|\alpha|-r} \subset \Sigma_0^{m+m'-r}. \tag{6.1.19}$$

Next we note that

$$a_1 := \partial_\xi^\alpha a \in \Gamma_r^{m-|\alpha|}, \quad b_1 := D_x^\alpha b \in \Gamma_{r-|\alpha|}^{m'}, \quad a_1 b_1 \in \Gamma_{r-|\alpha|}^{m+m'-|\alpha|}.$$

Therefore, Proposition 5.1.8 implies that

$$a_1 - \sigma_{a_1} \in \Gamma_0^{m-|\alpha|-r}, \quad b_1 - \sigma_{b_1} \in \Gamma_0^{m'-r+|\alpha|}, \quad a_1 b_1 - \sigma_{a_1 b_1} \in \Gamma_0^{m+m'-r}.$$

The first two properties imply that $a_1 b_1 - \sigma_{a_1}\sigma_{b_1} \in \Gamma_0^{m+m'-r}$ and therefore $\sigma_{a_1 b_1} - \sigma_{a_1}\sigma_{b_1} \in \Gamma_0^{m+m'-r}$. Combining with (6.1.19) implies that

$$\partial_\xi^\alpha \sigma_a D_x^\alpha \sigma_b - \sigma_{\partial_\xi^\alpha a D_x^\alpha b} \in \Sigma_0^{m+m'-r}$$

and the lemma follows, completing the proof of Theorem 6.1.1. □

6.1.3 A quantitative version

An examination of the proof of Theorem 6.1.1 yields estimates of the norm of the operator $T_a \circ T_b - T_{a\sharp b}$ in terms of the semi-norms (5.1.5) of the symbols a and b, for a given quantization T.

Theorem 6.1.4. *For all $s \in \mathbb{R}$, there is a constant C such that for $a \in \Gamma^m_r$, $b \in \Gamma^{m'}_r$ and $u \in H^{s+m+m'-r}(\mathbb{R}^d)$:*

$$\begin{aligned}
&\big\|T_a \circ T_b u - T_{a\sharp b} u\big\|_{H^s} \\
&\quad \le C\big(M^m_r(a;n)M^{m'}_0(b;n_0) + M^m_0(a;n)M^{m'}_r(b;n_0)\big)\ \big\|u\big\|_{H^{s+m+m'-r}}.
\end{aligned}$$

with $n_0 = [\frac{d}{2}] + 1$ and $n = n_0 + \overline{r}$.

Proof. We review the proof of Theorem 6.1.1.

a) In a first step, we may have to change the admissible cut-off so that the parameter ε_2 is smaller than $\frac{1}{4}$. Applying (5.1.25) and (5.1.26) implies that the norm of $T_a \circ T_b - T'_a \circ T'_b$ is bounded by

$$C\big(M^m_r(a;n_0)M^{m'}_0(b;n_0) + M^m_0(a;n_0)M^{m'}_r(b;n_0)\big).$$

Consider next for $|\alpha| < r$, $c_\alpha = \partial^\alpha_\xi a D^\alpha_x b$. For $s \ge 0$, the inequality

$$\big\|uv\big\|_{W^s} \le C\big(\big\|u\big\|_{L^\infty}\big\|v\big\|_{W^s} + \big\|u\big\|_{W^s}\big\|v\big\|_{L^\infty}\big) \tag{6.1.20}$$

implies that for symbols $p \in \Gamma^\mu_s$ and $q \in \Gamma^{\mu'}_s$,

$$M^{\mu+\mu'}_s(pq,n_0) \le C\big(M^\mu_s(p;n_0)M^{\mu'}_0(q;n_0) + M^\mu_0(p;n_0)M^{\mu'}_s(q;n_0)\big) \tag{6.1.21}$$

Moreover, for $s' \le s$, the interpolation inequality

$$\big\|u\big\|_{W^{s'}} \le C\big\|u\big\|^{1-\delta}_{L^\infty}\big\|u\big\|^\delta_{W^s} \tag{6.1.22}$$

with $\delta = \frac{s'}{s}$, implies that

$$M^\mu_{s'}(p,n) \le C\big(M^\mu_0(p;n)\big)^{1-\delta}\big(M^\mu_s(p;n)\big)^\delta.) \tag{6.1.23}$$

Therefore, by (6.1.21):

$$\begin{aligned}
M^{m+m'-|\alpha|}_{r-|\alpha|}(c_\alpha,n_0) \le C\big(&M^m_{r-|\alpha|}(a;n_0+|\alpha|)M^{m'}_{|\alpha|}(b;n_0) \\
&+ M^m_0(a;n_0+|\alpha|)M^{m'}_r(b;n_0)\big).
\end{aligned}$$

Using the interpolation inequality (6.1.22) for $M^m_{r-|\alpha|}(a; n_0+|\alpha|)$ and $M^{m'}_{|\alpha|}(b; n_0)$, we conclude that

$$M^{m+m'-|\alpha|}_{r-|\alpha|}(c_\alpha, n_0) \leq C\big(M^m_r(a; n)M^{m'}_0(b; n_0) + M^m_0(a; n)M^{m'}_r(b; n_0)\big) \tag{6.1.24}$$

Therefore, the norm of $T_{a\sharp b} - T'_{a\sharp b}$ is bounded by

$$C\big(M^m_r(a; n)M^{m'}_0(b; n_0) + M^m_0(a; n)M^{m'}_r(b; n_0)\big). \tag{6.1.25}$$

b) Assuming that the cut-off function satisfies the condition $\varepsilon_2 \leq \frac{1}{4}$, there holds $T_a \circ T_b = \sigma(x, D_x)$ with $\sigma = \sigma_a \sharp \sigma_b + q$ and $q \in \Sigma^{m+m'-r}_0$ satisfies

$$M^{m+m'-r}_0(q, n_0) \leq CM^m_0(a; n)M^{m'}_r(b; n_0). \tag{6.1.26}$$

Therefore, the norm of $q(x, D_x)$ is bounded by (6.1.25).

c) There holds

$$M^{m-r}_0(\partial^\alpha_\xi \sigma_a - \sigma_{\partial^\alpha_\xi a}; n_0) \leq CM^m_{r-|\alpha|}(a, n_0 + |\alpha|).$$

With the interpolation inequality (6.1.23), this implies that

$$M^{m+m'-r}_0(\partial^\alpha_\xi \sigma_a D^\alpha_x \sigma_b - \sigma_{\partial^\alpha_\xi a}\sigma_{D^\alpha_x b}; n_0)$$

is bounded by (6.1.25). Similarly, $M^{m+m'-r}_0(\sigma_{\partial^\alpha_\xi a}\sigma_{D^\alpha_x b} - \sigma_{\partial^\alpha_\xi a D^\alpha_x b}; n_0)$ satisfies a similar estimate, as well as $r = \sigma_a \sharp \sigma_b - \sigma_{a\sharp b}$. Therefore the norm of $r(x, D_x)$ is bounded by (6.1.25) and this finishes the proof of the theorem. □

6.2 Adjoints

6.2.1 The main result

When $p \in S^m_{1,1}$, the operator $P = p(x, D_x)$ maps $\mathscr{S}$ to $\mathscr{S}$. The adjoint operator P^* is therefore defined from $\mathscr{S}'$ to $\mathscr{S}'$, such that

$$\big\langle \big(p(x, D_x)\big)^* u, \overline{v}\big\rangle_{\mathscr{S}'\times\mathscr{S}} = \big\langle u, \overline{p(x, D_x)v}\big\rangle_{\mathscr{S}'\times\mathscr{S}}. \tag{6.2.1}$$

In general, P^* does not act from $\mathscr{S}$ to $\mathscr{S}$. However, this is true when the symbol p satisfies the spectral condition and in particular for paradifferential operators T_a.

Theorem 6.2.1. *Consider a matrix valued symbol $a \in \Gamma^m_r$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $a^*(x,\xi)$ the adjoint of the matrix $a(x,\xi)$. Then $(T_a)^* - T_b$ is of order $\leq m - r$ where*

$$b = \sum_{|\alpha|<r} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi a^*(x,\xi) \in \sum_{j<r} \Gamma^{m-j}_{r-j}. \tag{6.2.2}$$

It is sufficient to make the proof when a is scalar and from now on, we restrict ourselves to this case. Note that $D^\alpha_x \partial^\alpha_\xi \overline{a}(x,\xi)\Gamma^{m-|\alpha|}_{r-|\alpha|}$, so that the symbol b is well define.

In a preliminary step, we study the adjoint of operators with symbols in Σ^m_0.

Lemma 6.2.2. *Suppose that $p \in \Sigma^m_0$. Let $q \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ be defined by*

$$(\mathscr{F}_x q)(\eta,\xi) = (\mathscr{F}_x \overline{p})(\eta, \xi + \eta). \tag{6.2.3}$$

Then $q \in S^m_{1,1}$, satisfies the weak spectral condition (4.3.12) *and for all $u \in \mathscr{S}$:*

$$p(x,D_x)^* u = q(x,D_x)u. \tag{6.2.4}$$

Moreover, if p satisfies the spectral condition (4.2.16) *with parameter $\varepsilon < \frac{1}{2}$, then $q \in \Sigma^m_0$.*

Proof. Suppose first that $p \in \mathscr{S}$. We have checked in Lemma 4.2.3 that the adjoint of $p(x,D_x)$ is $q(x,D_x)$ with q given by (6.2.3), or equivalently

$$(\mathscr{F}_x q)(\eta,\xi) = \overline{(\mathscr{F}_x p)(-\eta, \xi + \eta)}.$$

If p satisfies the spectral condition (4.2.16) with parameter $\varepsilon < 1$, on the support of $\mathscr{F}_x q$, $|\eta| \leq \varepsilon(1 + |\xi + \eta|)$, implying that $|\xi| \leq |\xi + \eta| + |\eta| \leq (1+\varepsilon)(1 + |\xi+\eta|)$ and $(1-\varepsilon)|\eta| \leq \varepsilon(1+|\xi|)$. Therefore :

$$\begin{aligned} \mathrm{supp} \mathscr{F}_x q \subset \big\{ (\eta,\xi) \ : |\eta| \leq \frac{\varepsilon}{1-\varepsilon}(1+|\xi|) \\ \textit{and } 1 + |\xi+\eta| \geq \frac{1}{1+\varepsilon}|\xi| \big\}. \end{aligned} \tag{6.2.5}$$

This proves that q always satisfies the weak spectral condition (4.3.12) and the stronger form (4.2.16) when $\varepsilon < \frac{1}{2}$.

Let $\theta \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be such that for all α and β there is $C_{\alpha,\beta}$ such that

$$\left| \partial^\alpha_\eta \partial^\beta_\xi \theta(\eta,\xi) \right| \leq C_{\alpha,\beta}(1+|\xi|)^{-l\alpha|-|\beta|}, \tag{6.2.6}$$

(6.2.7) $$\theta = 1 \quad \text{on} \quad \mathrm{supp}\mathscr{F}_x q,$$

and, for some positive constants κ and λ:

(6.2.8) $$\theta(\eta, \xi) \ = 0 \quad \text{for } |\eta| \geq \kappa(1 + |\xi|) \quad \text{and for } \ 1 + |\xi \ + |\eta| \leq \lambda|\xi|$$

For instance, one can choose an admissible cut-off function ψ with parameters ε_1 and ε_1 such that $\varepsilon < \varepsilon_1 < \varepsilon 2 < 1$, so that $\psi = 1$ on the support of $\mathscr{F}_x\overline{p}$. Thus, $\theta(\eta, \xi) = \psi(\eta, \eta + \xi)$ satisfies (6.2.6), (6.2.7) and (6.2.8) with $\kappa = \frac{\varepsilon_2}{1-\varepsilon_2}$ and $\lambda = 1$.

Since $\theta = 1$ on the support of $\mathscr{F}_x q$, there holds

$$\begin{aligned} q(x, \xi) &= (2\pi)^{-d} \int e^{ix\eta}\theta(\eta, \xi)(\mathscr{F}_x\overline{p})(\eta, \xi + \eta)d\eta \\ &= (2\pi)^{-d} \int e^{i(x-y)\eta}\overline{p}(y, \xi + \eta)\theta(\eta, \xi)dyd\eta. \end{aligned}$$

Thus,

(6.2.9) $$q(x, \xi) = \int Q(x, y, \xi)dy$$

with

(6.2.10) $$Q(x, y, \xi) = (2\pi)^{-d} \int e^{-iy\eta}\overline{p}(x + y, \xi + \eta)\theta(\eta, \xi)d\eta.$$

By (6.2.7), $1+|\xi+|\eta| \approx 1+|\xi|$ on the support of θ. Thus, there are estimates of the form:

(6.2.11) $$\left|\partial_\eta^\alpha \overline{p}(x + y, \xi + \eta)\theta(\eta, \xi)\right| \leq C_\alpha(1 + |\xi|)^{m-\alpha}.$$

The integral in (6.2.10) is carried over the ball in η of radius $\kappa(1+|\xi|)$. Thus Lemma 4.3.2 implies

$$\left\|Q(x, \cdot, \xi)\right\|_{L^1} \leq CM_0^m(p, \tilde{d})(1 + |\xi|)^m$$

where we use the notations (5.1.5) for the semi-norms of symbols and $\tilde{d} = [\frac{d}{2}] + 1$. There are similar estimates for the ξ derivatives, implying that

(6.2.12) $$\left\|\partial_\xi^\beta q(x, \xi)\right\|_{L^1} \leq CM_0^m(p, |\beta| + \tilde{d})(1 + |\xi|)^{m-|\beta|}.$$

The estimates of the x-derivatives immediately follow using the spectral localization of $\mathscr{F}_x q$ in $\{|\eta| \leq \kappa|\xi|\}$.

Let $p \in \Sigma_0^m$. Consider approximations $p_n \in \mathscr{S}$ of p as indicated in Lemma 4.2.11. Then the estimates (6.2.12) imply that the symbols $q_n \in \mathscr{S}$ of the adjoints $p_n(x, D_x)^*$ are bounded in $S_{1,1}^m$, implying that $q_n \to q \in S_{1,1}^m$ and that $q(x, D_x)u = p(x, D_x)^*u$ when $u \in \mathscr{S}$. □

Together with Theorem 4.3.4, the lemma implies the following

Corollary 6.2.3. *If $p \in \Sigma_0^m$, the adjoints $p(x, D_x)^*$ maps $\mathscr{S}(\mathbb{R}^d)$ into itself and is an operator of order $\leq m$.*

End of the proof of Theorem 6.2.1.

a) We first check that the result does not depend on the choice of the cut-off function ψ used to define the operator T_a. Indeed, if ψ_1 and ψ_2 are two admissible cut-off functions, $T_a^{\psi_1} - T_a^{\psi_2} = p(x, D_x)$, where $p = \sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-r}$. Thus, Corollary 6.2.3 implies that $(T_a^{\psi_1})^* - (T_a^{\psi_2})^* = p(x, D_x)^*$ is of order $\leq m - r$.

Moreover, $D_x^\alpha \partial_\xi^\alpha \overline{a} \in \Gamma_{r-|\alpha|}^{m-|\alpha|}$ and thus $T^{\psi_1}_{D_x^\alpha \partial_\xi^\alpha \overline{a}} - T^{\psi_2}_{D_x^\alpha \partial_\xi^\alpha \overline{a}}$ is of order $\leq m - |\alpha| - (r - |\alpha|) = m - r$. Hence, $T_b^{\psi_1} - T_b^{\psi_2}$ is of order $\leq m - r$.

Therefore, we can assume that the function ψ satisfies the localization property (5.1.6) with $\varepsilon_2 < \frac{1}{2}$.

b) By Lemma 6.2.2, $(T_a^\psi)^* = q(x, D_x)$ where the symbol q is given by (6.2.9) (6.2.10) with $p = \sigma_a^\psi$.

Introduce an admissible cut-off function $\theta(\eta, \xi)$ equal to 1 on $\{|\eta| \leq 2\varepsilon(1 + |\xi|)$ and vanishing on $\{|\eta| \geq \varepsilon_2'(1 + |\xi|)$ where $2\varepsilon < \varepsilon_2' < 1$. Then, θ satisfies (6.2.6) and (6.2.8) with $\kappa = \varepsilon_2'$ and $\lambda = \frac{1}{1-\varepsilon_2'}$. Moreover,

$$\theta = 1 \quad \text{on } \mathrm{supp}\mathscr{F}_x\overline{\sigma_a} \quad \text{and on } \mathrm{supp}\mathscr{F}_x q. \tag{6.2.13}$$

We use Taylor expansions:

$$\overline{\sigma_a}(x + y, \xi + \eta) = \sum_{|\alpha|<r} \frac{1}{\alpha!}\eta^\alpha \partial_\xi^\alpha \overline{\sigma_a}(x + y, \xi) + \sum_{|\alpha|=\overline{r}} \eta^\alpha \rho_\alpha(x + y, \xi, \alpha),$$

where $\overline{r}$ is the smallest integer $\geq r$. Substituting in (6.2.9) yields

$$Q(x, y, \xi) = \sum_{|\alpha|<r} \frac{1}{\alpha!} Q_\alpha(x, y, \xi) + \sum_{|\alpha|=\overline{r}} R_\alpha(x, y, \xi)$$

with

$$Q_\alpha(x, y, \xi) = (2\pi)^{-d} \int e^{-iy\eta} \partial_\xi^\alpha \overline{\sigma_a}(x + y, \xi)\eta^\alpha \theta(\eta, \xi) d\eta,$$

$$R_\alpha(x, y, \xi) = (2\pi)^{-d} \int e^{-iy\eta} \rho_\alpha(x + y, \eta, \xi)\eta^\alpha \theta(\eta, \xi) d\eta.$$

The ρ_α are given by integrals of $\partial_\xi^\alpha \overline{\sigma_a}(x+y, \xi+t\eta)$ over $t \in [0, 1]$. The support condition implies that $1 + |\xi + t\eta| \approx 1 + |\xi|$ on the support of θ, so that the

$\rho_\alpha \theta$ satisfy estimates analogous to (6.2.11), implying by Lemma 4.3.2 that $R_\alpha(x, \cdot \xi) \in L^1$. There are similar estimates for the y-derivatives, implying that one can perform integrations by parts and thus

$$r_\alpha(x,\xi) := \int R_\alpha(x,y,\xi)dy = \int \widetilde{R}_\alpha(x,y,\xi)dy \tag{6.2.14}$$

with

$$\widetilde{R}_\alpha(x,y,\xi) = (2\pi)^{-d} \int e^{-iy\eta} D_x^\alpha \rho_\alpha(x+y,\eta,\xi)\theta(\eta,\xi)d\eta.$$

By Proposition 5.1.9, for $|\alpha| = \overline{r} \geq r$, $\partial_\xi^\alpha \sigma_a \in \Sigma^{m-|\alpha|}$ and $D_x^\alpha \partial_\xi^\alpha \sigma_a \in \Sigma^{m-|\alpha|-r+|\alpha|} = \Sigma_0^{m-r}$. This implies that $D_x^\alpha \rho_\alpha(x+y,\eta,\xi)\theta(\eta,\xi)$ satisfies estimates of the form

$$\left|\partial_\eta^\beta \big(D_x^\alpha \rho_\alpha(x+y,\eta,\xi)\theta(\eta,\xi)\big)\right| \leq C_\beta(1+|\xi|)^{m-r-|\beta|}.$$

With Lemma 4.3.2, we conclude that

$$\left|r_\alpha(x,\xi)\right| \leq C(1+|\xi|)^{m-r}.$$

There are similar estimates for the ξ derivatives, and therefore $r = \sum r_\alpha$ satisfies estimates of the form

$$\left|\partial_\xi^\beta r(x,\xi)\right| \leq C_\beta(1+|\xi|)^{m-r-|\beta|}. \tag{6.2.15}$$

Similarly, since $\theta = 1$ on the support of $\mathscr{F}_x \overline{p}$, there holds

$$q_\alpha(x,\xi) := \int Q_\alpha(x,y,\xi)dy = D_x^\alpha \partial_\xi^\alpha \overline{\sigma_a}(x,\xi). \tag{6.2.16}$$

Summing up, we have proved that

$$q(x,\xi) = \sum_{|\alpha|<r} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{\sigma_a}(x,\xi) + r(x,\xi). \tag{6.2.17}$$

The spectral property is satisfied by q and $\overline{\sigma_a}$, thus by r, and therefore (6.2.16) implies that $r \in \Sigma_0^{m-r}$, so that $r(x,D_x)$ is of order $\leq m-r$.

c) Because $D_x^\alpha \overline{a} \in \Gamma_{r-|\alpha|}^m$, Proposition 5.1.10 implies that $r_1 = D_x^\alpha \partial_\xi^\alpha \overline{\sigma_a} - \sigma_{D_x^\alpha \partial_\xi^\alpha \overline{a}}$ belongs to $\Sigma_0^{m-|\alpha|-(r-|\alpha|)} = \Sigma_0^{m-r}$. Thus $r_1(x,D_x)$ is also of order $\leq m-r$. Adding up, we have proved that $(T_a)^* - T_b = r_1(x,D_x) + r(x,D_x)$ is order $\leq m-r$. □

Theorem 6.2.1 also admits a quantitative version:

Theorem 6.2.4. *For all $s \in \mathbb{R}$, there is a constant C such that for all $a \in \Gamma_r^m$ and $u \in H^{m+s-r}$, there holds*

$$\big\| (T_a)^* u - T_b u \big\|_{H^s} \leq C M_r^m(a;n) \, \|u\|_{H^{s+m-r}}.$$

with $n_0 = [\frac{d}{2}] + 1$, $n = n_0 + \overline{r}$ and b given by (6.2.2).

6.3 Applications

6.3.1 Elliptic estimates

Definition 6.3.1. *A scalar symbol $a \in \Gamma_0^m$ is said to be elliptic if there is constant $c > 0$ such that*

$$\forall (x,\xi) : \quad |a(x,\xi)| \geq c(1+|\xi|)^m \tag{6.3.1}$$

More generally, a $N \times N$ matrix valued symbol $a \in \Gamma_0^m$ is said to be elliptic if $\det a \in \Gamma_0^{Nm}$ is elliptic.

For matrices, the condition is equivalent to the property that $a(x,\xi)$ is invertible for all (x,ξ) and there is a constant C such that

$$\forall (x,\xi) : \quad |a^{-1}(x,\xi)| \leq C(1+|\xi|)^{-m}. \tag{6.3.2}$$

The following result is elementary:

Lemma 6.3.2. *If a is an elliptic symbol in Γ_r^m, then a^{-1} is a symbol in Γ_r^{-m}.*

Theorem 6.3.3. *Given s and m, there ares constants C_0 and C_1 such that for all elliptic symbol $a \in \Gamma_1^m$ and $u \in H^s$:*

$$\|u\|_{H^s} \leq K_0 \, \|T_a u\|_{H^{s-m}} + K_1 \|u\|_{H^{s-1}}, \tag{6.3.3}$$

with

$$K_0 = C_0 M_0^{-m}(a^{-1}; n_0),$$
$$K_1 = C_1 \big(M_1^{-m}(a^{-1}; n_0+1) M_0^m(a;n_0) + M_0^{-m}(a^{-1}; n_0+1) M_1^m(a;n_0) \big).$$

and $n_0 = [\frac{d}{2}] + 1$.

Proof. Let $b = a^{-1}$. For $r = 1$, the symbolic composition $b \sharp a$ reduces to the product $ba = 1$. Therefore $u = T_b T_a u + Ru$ where R is of order -1, and the precise estimate (6.3.3) follows from (5.1.25) and Theorem 6.1.4. $\square$

6.3.2 Gårding's inequality

Theorem 6.3.4. *There are constants C_0, C_1 and C_2 such that for all $N \times N$ matrix symbol $a \in \Gamma_1^m$ satisfying*

$$\forall (x,\xi) \ : \quad \operatorname{Re} a(x,\xi) \geq c(1+|\xi|)^m \mathrm{Id} \tag{6.3.4}$$

for some constant $c > 0$, the positive square root $b = (\operatorname{Re} a)^{\frac{1}{2}}$ belongs to $\Gamma_1^{m/2}$ and is elliptic and for all $u \in H^m(\mathbb{R}^d)$: and

$$\frac{1}{K_0^2} \|u\|^2_{H^{m/2}} \leq \operatorname{Re} \big(T_a u, u\big)_{L^2} + (K_1 + K_2^2) \|u\|^2_{H^{\frac{m-1}{2}}}. \tag{6.3.5}$$

with

$$\begin{aligned}
K_0 &= M_0^{-m/2}(b^{-1}; n_0), \\
K_1 &= C_1 \big(M_1^m(a; n_0+1) + M_1^{m/2}(b; n_0+1) M_0^{m/2}(b; n_0+1) \big) \\
K_2 &= \frac{C_2}{K_0} \big(M_1^{m/2}(b^{-1}; n_0+1) M_0^{m/2}(b; n_0) + M_0^{-m/2}(b^{-1}; n_0+1) M_1^{m/2}(b; n_0) \big).
\end{aligned}$$

Proof. There holds

$$\operatorname{Re} \big(T_a u, u\big)_{L^2} = \big(\operatorname{Re} T_a u, u\big)_{L^2}, \qquad \operatorname{Re} T_a = \frac{1}{2}(T_a u + (T_a)^*).$$

By Theorem 6.2.1, $T_a u + (T_a)^* = 2T_{\operatorname{Re} a} + R$ where R is of order $m-1$.

The assumption is that $\operatorname{Re} a$ is elliptic and definite positive. It implies that its positive square root $b = b^* = (\operatorname{Re} a)^{\frac{1}{2}}$ is an elliptic symbol in $\Gamma_1^{m/2}$. Therefore $T_{\operatorname{Re} a} = (T_b)^2 T_b + R'$ where R' is of order $m-1$. Thus :

$$\operatorname{Re} \big(T_a u, u\big)_{L^2} = \big\|T_b u\big\|^2_{L^2} + \big((R+R')u, u\big)_{L^2}.$$

We conclude by applying the elliptic estimate of Theorem 6.3.3 to T_b and together with the estimate

$$\big|\big((R+R')u, u\big)_{L^2}\big| \leq C \big\|u\big\|^2_{H^{\frac{m-1}{2}}}$$

where C is the norm of $R + R'$ from $H^{\frac{m-1}{2}}$ to $H^{\frac{1-m}{2}}$. □

6.4 Pluri-homogeneous calculus

In applications to PDE's, the notion of *principal symbol* plays a crucial role. This leads to consider very naturally *homogenenous* symbols, which are not C^∞ at $\xi = 0$ except when they are polynomials. In this section, we briefly indicated how the calculus developed above is easily adapted to this framework. The main idea is that low frequencies are irrelevant in the smoothness analysis and only contribute to remainders.

Definition 6.4.1 (Homogeneous symbols)**.** *For $m \in \mathbb{R}$ and $r \geq 0$, $\dot\Gamma^m_r$ is the set of functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d\backslash\{0\})$ which are homogeneous of degree m and C^∞ with respect to $\xi \neq 0$, and such that for all $\alpha \in \mathbb{N}^d$ and $\xi \neq 0$, $\partial_\xi^\alpha a(\,\cdot, ,\xi) \in W^r(\mathbb{R}^d)$ and*

$$\sup_{|\xi|=1} \left\| \partial_\xi^\alpha a(\,\cdot\,,\xi) \right\|_{W^r} \; < \; +\infty. \tag{6.4.1}$$

The following lemma is elementary.

Lemma 6.4.2. *Let $\theta \in C^\infty(\mathbb{R}^d)$ be such that*

$$1 - \theta \in C_0^\infty(\mathbb{R}^d) \qquad \textit{and} \quad \theta = 0 \quad \textit{in a neighborhood of } 0. \tag{6.4.2}$$

Then for all $a \in \dot\Gamma^m_r$, the symbol $a(x,\xi)\theta(\xi)$ belongs to Γ^m_r.

If θ' is another function satisfying (6.4.2), then $a\theta - a\theta' \in \Gamma^\mu_r$ for all $\mu \in \mathbb{R}$.

Then one can define the following quantization for $a \in \dot\Gamma^m_r$:

$$\dot T_a u = T_{a\theta} u. \tag{6.4.3}$$

Then, $\dot T_a$ is of order m, and changing the function θ modifies $\dot T_a$ by an operator of order $-\infty$, that is of order $\leq \mu$ for all μ, and thus infinitely smoothing. Similarly, if $a \in \Gamma^m_r$ then $T_a - \dot T_a$ is of order $-\infty$

The rules for composition or taking adjoints leads to consider sums of homogeneous symbols of different degrees.

Definition 6.4.3 (Pluri-homogeneous symbols)**.** *For $m \in \mathbb{R}$ and $r > 0$, $\widetilde\Gamma^m_r$ is the space of sums*

$$a = \sum_{j<r} a_j \tag{6.4.4}$$

with $a_j \in \dot\Gamma^{m-j}_{r-j}$.

The operator $\dot T_a = \sum \dot T_{a_j}$ is still defined by (6.4.3). Note that $\dot T_{a_j}$ is uniquely defined up to an operator of order $\leq m - j - (r-j) = m - r$ so that $\dot T_a$ is independent of the cut-off functions modulo operators of order $\leq m - r$.

Then Theorems 6.1.1 and 6.2.1 have the following extensions:

Theorem 6.4.4. *Consider* $a = \sum a_j \in \widetilde{\Gamma}^m_r$ *and* $b = \sum b_j \in \widetilde{\Gamma}^{m'}_r$, *where* $r > 0$. *Then*

$$a\sharp b := \sum_{j<r} \sum_{|\alpha|+k+l=j} \frac{1}{\alpha!} \partial^\alpha_\xi a_k(x,\xi)\, D^\alpha_x b_l(x,\xi) \tag{6.4.5}$$

belongs to $\widetilde{\Gamma}^{m+m'}_r$. *Moreover,* $\dot{T}_a \circ \dot{T}_b - \dot{T}_{a\sharp b}$ *is of order* $\leq m + m' - r$.

Theorem 6.4.5. *Consider* $a = \sum a_j \in \widetilde{\Gamma}^m_r$. *Then*

$$b = \sum_{j<r} \sum_{|\alpha|+k=j} \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi a^*_k(x,\xi) \tag{6.4.6}$$

belongs to $\widetilde{\Gamma}^m_r$. *and* $(\dot{T}_a)^* - \dot{T}_b$ *is of order* $\leq m - r$.

Part III

Applications

Chapter 7

Nonlinear Hyperbolic Systems

In this chapter, we prove the local well posedness of the Cauchy problem for nonlinear symmetrizable hyperbolic systems. The main result of the chapter is Theorem 7.3.3. We also prove a blow-up criterion : the life span of the smooth solution is limited either by a blow up in L^∞ or the apparition of a singularity in the gradient of the solution. In one space dimension, for systems of conservation laws, there are global existence theorems in spaces of functions with bounded variation, and singularities (shocks) do appear in general. The existence of an analogous result in multi-D is a completely open question. Possible references for the hyperbolic Cauchy problem are [Fr1, Fr2, Fr3] in the linear case, [Gå, La, Miz, Ma1, Ma2, Ma3, Hör, Tay] for nonlinear equations or systems. In particular, this chapter is clearly inspired by the work of made by A.Majda ([Ma1, Ma2] who applied the pseudo-differential calculus to nonlinear boundary value problems and shock waves. As said in the introduction, the present chapter is a preparation for this kind of developments.

The main step consists in proving *a priori* energy estimates. The proof is classical for systems which are symmetric in the sense of Friedrichs, using integrations by parts as shown in Section 3.3. Here we assume only *microlocal symmetrizability*, property which is satisfied for instance for strictly hyperbolic systems or more generally for hyperbolic systems with constant multiplicity. As explained ind Chapter 3, for constant coefficient systems, the symmetrizers are Fourier multipliers. The constant coefficient analysis also provides us with symbols which symmetrize the symbol of nonconstant coefficient systems. We use the para-differential calculus to transform these symbolic symmetrizers into operators which actually symmetrize the PDE systems.

7.1 The L^2 linear theory

7.1.1 Statement of the result

We consider the Cauchy problem

$$\begin{cases} Lu = f, & \text{on } \;]0,T[\times\mathbb{R}^d \\ u_{|t=0} = h \end{cases} \tag{7.1.1}$$

for a first order $N \times N$ system with variable coefficients

$$Lu := \partial_t u + \sum_{j=1}^{d} A_j(t,x)\partial_j u \tag{7.1.2}$$

Assumption 7.1.1. *The matrices A_j have $W^{1,\infty}$ coefficients on $[0,T]\times\mathbb{R}^d$*

The symbol of the equation is

$$A(t,x,\xi) = \sum_{j=1}^{d} \xi_j A_j(t,x) \tag{7.1.3}$$

The system is assumed to be *hyperbolic* in the following sense :

Assumption 7.1.2 (Microlocal symmetrizability)**.** *There is a $N\times N$ matrix $S(t,x,\xi)$, homogeneous of degree 0 in ξ, with entries C^∞ in $\xi \neq 0$ and $W^{1,\infty}$ in $(t,x) \in [0,T]\times\mathbb{R}^d$ such that:*

i) $S(t,x,\xi)$ is self adjoint and definite positive, and there is $c > 0$ such that for all (t,x,ξ), $S(t,x,\xi) \geq c\mathrm{Id}$;

ii) For all (t,x,ξ), $S(t,x,\xi)A(t,x,\xi)$ is self-adjoint.

It means that

i) for all fixed $(\underline{t},\underline{x})$, the constant coefficient system $\partial_t + A(\underline{t},\underline{x},\partial_x)$ is strongly hyperbolic and admits a symmetrizer $S(\underline{t},\underline{x},D_x)$,

ii) the symbol $S(\underline{t},\underline{x},\xi)$ is $W^{1,\infty}$ in $(\underline{t},\underline{x})$.

In particular, the case of symmetric systems in the sense of Friedrichs corresponds to the case where $S(t,x,\xi)$ can be chosen independent of ξ (see Assumption 3.3.2).

Theorem 7.1.3. *For $f \in L^1([0,T]; L^2(\mathbb{R}^d))$ and $h \in L^2(\mathbb{R}^d))$ the Cauchy problem (7.1.1) has a unique solution $u \in C^0([0,T]; L^2(\mathbb{R}^d))$. Moreover, here are constants C and K such that for all f and h the solution u satisfies :*

$$\|u(t)\|_{L^2} \le Ce^{Kt}\|u(0)\|_{L^2} + C\int_0^t Ce^{K(t-s)}\|Lu(s)\|_{L^2} ds. \tag{7.1.4}$$

The precise dependence of C and K on bounds for the coefficients A_j and S is given in (7.1.26) and (7.1.27) below. Knowing this dependence is crucial for the nonlinear theory.

The proof of this Theorem is in three steps:

- There are constants C and K such that the estimate (7.1.4) is satisfied for all $u \in H^1([0,T] \times \mathbb{R}^d)$. For this, the main idea is to compare the operator L to its para-differential version $\partial_t + T_{iA}$.
- We show that if $u \in L^2([0,T] \times \mathbb{R}^d)$ is a solution of the Cauchy problem, with $f \in L^1([0,T]; L^2)$ and $h \in L^2$, then actually $u \in C^0([0,T]; L^2)$ and satisfies the energy estimate (7.1.4). This implies uniqueness. The proof is based on a regularization process (Friedrich's lemma) which we prove here using again the para-differential calculus.
- We construct a solution $u \in L^2([0,T] \times \mathbb{R}^d)$ of the Cauchy problem, which by step 2 is actually in $C^0([0,T]; L^2)$.

7.1.2 Paralinearisation

We use the paradifferential calculus on $\mathbb{R}^d$ which applies to functions and symbols on $\mathbb{R}^d$. Below, we assume that the quantization is fixed, that is associated to a fixed admissible cut-off function ψ and we use the notation T_a for para-products or para-differential operators. We extend this calculus to symbols and functions which also depend on time : When a and u are symbols and functions on $[0,T] \times \mathbb{R}^d$, we still denote by $T_a u$ the spatial para-differential operator (or para-product) such that for all $t \in [0,T]$

$$(T_a u)(t, \cdot\,) = T_{a(t, \cdot\,)}\, u(t, \cdot\,)\,. \tag{7.1.5}$$

Accordingly, we use the following notations:

Definition 7.1.4. $\Gamma^m_k([0,T] \times \mathbb{R}^d)$ *denotes the space of symbols $a(t,x,\xi)$ such that the mapping $t \mapsto a(t, \cdot\,)$ is bounded from $[0,T]$ into the space $\Gamma^m_k(\mathbb{R}^d)$ of Definition 5.1.2.*

Similarly, $\dot\Gamma^m_k([0,T] \times \mathbb{R}^d)$ denotes the space of symbols $a(t,x,\xi)$, homogeneous of degree m in $\xi \ne 0$, such that the mapping $t \mapsto a(t, \cdot\,)$ is bounded from $[0,T]$ into the space $\dot\Gamma^m_k(\mathbb{R}^d)$ of Definition 6.4.1.

In particular, the assumptions on the coefficients of A_j imply that the symbol $A(t,x,\xi)$ belongs to $\dot\Gamma^1_1([0,T]\times\mathbb{R}^d)$ and also to $\Gamma^1_1([0,T]\times\mathbb{R}^d)$.

Notations. Introduce

$$M_0(A)=\sum_j \big\|A_j\big\|_{L^\infty([0,T]\times\mathbb{R}^d)}, \tag{7.1.6}$$

$$M_1(A)=\sum_j \sup_{t\in[0,T]} \big\|A_j(t,\cdot)\big\|_{W^{1,\infty}(\mathbb{R}^d)}. \tag{7.1.7}$$

By definition $iA(t,x,\xi)=\sum A_j(t,x)(i\xi_j)$ and

$$T_{iA}v=\sum_{j=1}^d T_{A_j}\partial_{x_j}v. \tag{7.1.8}$$

Introduce

$$Rv:=A(t,x,\partial_x)v-T_{iA}v=\sum_{j=1}^d \big(A_j-T_{A_j}\big)\partial_{x_j}v. \tag{7.1.9}$$

Theorem 5.2.9 implies the following lemma:

Lemma 7.1.5. *There is a constant γ such that for all $u\in C^0([0,T];L^2(\mathbb{R}^d))$ there holds for almost all $t\in[0,T]$:*

$$\big\|A(t,x,\partial_x)u(t)-T_{iA}u(t)\big\|_{L^2}\le\gamma M_1(A)\big\|u(t)\big\|_{L^2}. \tag{7.1.10}$$

Using Gronwall's lemma, this lemma implies that the estimate (7.1.4) is a consequence of similar estimates where Lu replaced by $\partial_t u+T_{iA}u$.

7.1.3 Symmetrizers

The Assumption 7.1.2 provides us with symbols S such that

$$S\in\ \dot\Gamma^0_1([0,T]\times\mathbb{R}^d),\qquad \partial_t S\in\ \dot\Gamma^0_0([0,T]\times\mathbb{R}^d). \tag{7.1.11}$$

The general idea is to show that the para-differential operator T_S is a symmetrizer for the equation. However, we have to take care of the singularity of S at $\xi=0$ and also of remainders which will occur in the symbolic calculus. This leads to technical modifications. First, note that the ellipticity assumptions on S imply that

$$S^{-1},\ S^{\pm\frac12}\in\dot\Gamma^0_1([0,T]\times\mathbb{R}^d). \tag{7.1.12}$$

Introduce $\theta \in C^\infty(\mathbb{R}^d)$ such that $0 \le \theta \le 1$, $\theta = 0$ for $|\xi| \ge 2$ and such that $\theta = 1$ for all $|\xi| \le 1$. With a parameter $\lambda \ge 1$ to be chosen let

$$V(t,x,\xi) = S^{\frac{1}{2}}(t,x,\xi)(1-\theta(\lambda^{-1}\xi)) \tag{7.1.13}$$

$$\Sigma(t) = \big(T_{V(t)}\big)^* T_{V(t)} + \big(\theta(\lambda^{-1}D_x)\big)^2 \mathrm{Id} \tag{7.1.14}$$

In particular,

$$\big(\Sigma u, u\big)_{L^2} = \big\|T_V u\big\|^2_{L^2} + \big\|\theta(\lambda^{-1}D_x)u\big\|^2_{L^2}. \tag{7.1.15}$$

To control the norm of various operators defined with S, we use the following semi-norms (see (5.1.5)):
Notations. For $r \ge 0$ and P in $\Gamma^0_k([0,T]\times\mathbb{R}^d)$ and Q in $\dot\Gamma^0_k([0,T]\times\mathbb{R}^d)$,let

$$M_r(P;n) = \sup_{|\alpha|\le n} \sup_{(t,\xi)\in[0,T]\times\mathbb{R}^d} (1+|\xi|)^{|\alpha|} \big\|\partial^\alpha_\xi P(t,\,\cdot\xi)\big\|_{W^r(\mathbb{R}^d)}. \tag{7.1.16}$$

$$\dot M_r(Q;n) = \sup_{|\alpha|\le n} \sup_{(t,\xi)\in[0,T]\times S^{d-1}} \big\|\partial^\alpha_\xi P(t,\,\cdot\xi)\big\|_{W^r(\mathbb{R}^d)}. \tag{7.1.17}$$

In particular, for $k \in \mathbb{N}$,

$$M_k(P;n) = \sup_{|\alpha|\le n} \sup_{|\beta|\le k} \big\|(1+|\xi|)^{|\alpha|}\partial^\beta_x\partial^\alpha_\xi P(t,.,.)\big\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)}. \tag{7.1.18}$$

There is a similar expression for $\dot M_k(Q;n)$.

Lemma 7.1.6. *There are constants γ_0, γ_1 and γ such that for all $t \in [0,T]$, the operator $\Sigma(t)$ satisfies for all $v \in L^2(\mathbb{R}^d)$:*

$$\big\|\Sigma(t)v\big\|_{L^2} \le C\big\|v\big\|_{L^2}. \tag{7.1.19}$$

$$\big\|v\big\|^2_{L^2} \le C_0\big(\Sigma(t)v, v\big)_{L^2} \tag{7.1.20}$$

where

$$C = \gamma\big(\dot M_0(S^{\frac{1}{2}};n)\big)^2 + 1, \qquad C_0 = \gamma_0\big(\dot M_0(S^{-\frac{1}{2}};n)\big)^2 \tag{7.1.21}$$

with $n = [\frac{n}{2}] + 2$, provided that $\lambda \ge \max\{\lambda_1, 2\}$ with

$$\lambda_1 = \gamma_1\Big(\dot M_0(S^{-\frac{1}{2}};n)\dot M_1(S^{\frac{1}{2}};n) + \dot M_1(S^{-\frac{1}{2}};n)\dot M_0(S^{\frac{1}{2}};n)\Big). \tag{7.1.22}$$

Proof. By (5.1.25),

$$\|T_{V(t)}v\|_{L^2} \leq \gamma M_0(V;n_0)\|v\|_{L^2},$$

with $n_0 = [\frac{d}{2}] + 1$. Thus the same estimate is valid for $(T_{S_0})^*$ and therefore

$$\|(T_{V(t)})^* T_{V(t)}v\|_{L^2} \leq \gamma^2 \big(M_0(V;n_0)\big)^2 \|v\|_{L^2}.$$

Next, we remark that

$$M_0(V;n_0) \leq \gamma' \dot{M}_0(S^{\frac{1}{2}};n_0)$$

with γ' independent of λ. This implies (7.1.19)

Let $V_1 = S^{\frac{1}{2}}(1-\theta)$ and $W = S^{-\frac{1}{2}}(1-\theta)$. Then, for $\lambda \geq 2$:

$$V = V_1(1-\theta(\lambda^{-1}\xi)), \qquad W_1 V_1 = (1-\theta)^2 \mathrm{Id}.$$

Therefore Theorem 6.1.1 implies that

$$T_{W_1}T_V = (\mathrm{Id} + R)(1-\theta(\lambda^{-1}D_x))$$

where R is of order ≤ -1. More precisely, with (5.1.25) and Theorem 6.1.4, we see that

$$\|(1-\theta(\lambda^{-1}D_x))\ v\|_{L^2} \leq \gamma_0 M_0(W_1;n)\ \|T_V v\|_{L^2} + C_1\|(1-\theta(\lambda^{-1}D_x))\|_{H^{-1}}$$

with

$$\begin{aligned} C_1 &= \gamma_1\Big(M_0(W_1;n)M_1(V_1;n) + M_1(V_1;n)M_0(W_1;n)\Big). \\ &\leq \gamma_1'\Big(\dot{M}_0(S^{-\frac{1}{2}};n)\dot{M}_1(S^{\frac{1}{2}};n) + \dot{M}_1(S^{-\frac{1}{2}};n)\dot{M}_0(S^{\frac{1}{2}};n)\Big) := \frac{1}{2}\lambda_1. \end{aligned}$$

Because

$$\|(1-\theta(\lambda^{-1}D_x))\|_{H^{-1}} \leq \lambda^{-1}\|(1-\theta(\lambda^{-1}D_x))\|_{L^2},$$

this implies that for $\lambda \geq \lambda_1$, there holds

$$\begin{aligned} \|(1-\theta(\lambda^{-1}D_x))\ v\|_{L^2} &\leq 2\gamma_0 \dot{M}_0(W_1;n)\ \|T_V v\|_{L^2} \\ &\leq \gamma_0' \dot{M}_0(S^{-\frac{1}{2}};n)\ \|T_V v\|_{L^2}. \end{aligned}$$

Squaring and this implies

$$\|v\|^2_{L^2} \leq \gamma_0\big(M_0(S^{-\frac{1}{2}};n)\big)^2\ \|T_V v\|^2_{L^2} + \|\theta(\lambda^{-1}D_x)v\|^2_{L^2}.$$

Using (7.1.15), the estimate (7.1.20) follows. □

From now on we suppose that λ is fixed and equal to $\max\{2, \lambda_1\}$.

Lemma 7.1.7. *There is a constant γ_2 such that for all $t \in [0,T]$ and $v \in H^1(\mathbb{R}^d)$:*

$$-\frac{1}{2}\big(\partial_t \Sigma v, v\big)_{L^2} + \mathrm{Re}\,\big(\Sigma(t) T_{iA} v, v\big)_{L^2} \leq K \big\|u\big\|^2_{L^2} \tag{7.1.23}$$

with

$$\begin{aligned} K = \gamma_2\big(& \max\{2, \lambda_1\} M_0(A) + \dot{M}_0(\partial_t S^{\frac{1}{2}}; n) \dot{M}_0(S^{\frac{1}{2}}; n) \\ & + \dot{M}_1(S^{\frac{1}{2}}; n) \dot{M}_0(S^{\frac{1}{2}}) M_0(A) + \dot{M}_0(S; n) M_1(A)\big). \end{aligned} \tag{7.1.24}$$

Proof. The definition of the quantization T implies that $\partial_t T_V = T_{\partial_t V}$. Thus $\partial_t \Sigma = (T_{\partial_t V}) * T_V + (T_V)^* T_{\partial_t V}$ is of order 0 and contributes to the first term in K.

The symbolic calculus implies that

$$(T_V)^* T_V T_{iA} - T_{iSA(1-\theta_\lambda)^2}$$

is of order ≤ 0, where $\theta_\lambda(\xi) = \theta(\lambda^{-1}\xi)$. Since $\mathrm{Re}\, iSA = 0$, it follows that $\mathrm{Re}\, T_{iSA(1-\theta_\lambda)^2}$ is also of order ≤ 0. More precisely, there holds

$$\big\|\mathrm{Re}\,\big((T_V)^* T_V T_{iA}\big) u\big\|_{L^2} \leq K_2 \big\|u\big\|_{L^2}$$

with

$$K_2 = \gamma_2\Big(\dot{M}_1(S^{\frac{1}{2}}; n) \dot{M}_0(S^{\frac{1}{2}}) M_0(A) + \dot{M}_0(S; n) M_1(A)\Big).$$

In addition,

$$\big\|\theta^2_\lambda(D_x) T_{iA}) u\big\|_{L^2} \leq 2\lambda_1 \big\|T_{iA}) u\big\|_{H^{-1}} \gamma_2' M_0(A) \big\|u\big\|_{L^2}.$$

This implies (7.1.23). □

7.1.4 The basic L^2 estimate

Proposition 7.1.8. *There are constants C and K such that for all $u \in L^2([0,T]; H^1(\mathbb{R}^d))$ with $\partial_t u \in L^1([0,T]; L^2(\mathbb{R}^d))$*

$$\big\|u(t)\big\|_{L^2} \leq C e^{Kt} \big\|u(0)\big\|_{L^2} + C \int_0^t e^{K(t-t')} \big\|Lu(t')\big\|_{L^2} dt'. \tag{7.1.25}$$

Moreover, there are functions $\mathcal{C}$ and $\mathcal{K}$ n such that the constants C and K are of the form

$$C = \mathcal{C}\big(M_0(S^{-\frac{1}{2}}; n), M_0(S^{\frac{1}{2}}; n)\big) \tag{7.1.26}$$

$$K = \mathcal{K}\big(M_0(S^{-\frac{1}{2}};n), M_0(S^{\frac{1}{2}};n), M_0(A)\big) \big(M_0(\partial_t S^{\frac{1}{2}};n) + M_1(S^{\frac{1}{2}};n) + M_1(S^{-\frac{1}{2}};n) + M_1(A)\big) \tag{7.1.27}$$

We first prove a similar estimate for the para-differential equation

$$\partial_t u + T_{iA} u = f.$$

Proposition 7.1.9. *There are constants* C *and* K *as in* (7.1.26) *and* (7.1.27) *such that for all* $u \in L^2([0,T]; H^1(\mathbb{R}^d))$ *with* $\partial_t u \in L^1([0,T]; L^2(\mathbb{R}^d))$,

$$\big\|u(t)\big\|_{L^2} \le Ce^{Kt}\big\|u(0)\big\|_{L^2} + C\int_0^t e^{K(t-t')}\big\|(\partial_t u + T_{iA}u)(t')\big\|_{L^2} dt'. \tag{7.1.28}$$

Proof. This is an application of the method exposed in Section 3.1. When u is smooth (for instance in $C^1([0,T]; H^1)$, there holds

$$\partial_t \big(\Sigma u, u\big)_{L^2} = 2\mathrm{Re}\,\big(\Sigma f, u\big)_{L^2} + \mathrm{Re}\,\big((\partial_t \Sigma - 2\Sigma T_{iA})u, u\big)_{L^2}$$

with $f = \partial_t u + T_{iA}u$. Thus $\mathcal{E} = (\Sigma u, u)_{L^2}$ satisfies

$$\partial_t \mathcal{E} \le 2\mathcal{E}^{\frac{1}{2}}\mathcal{F}^{\frac{1}{2}} + 2KC_0\mathcal{E}$$

where $\mathcal{F} = (\Sigma f, f)_{L^2}$ and K and C_0 given by (7.1.24) and (7.1.21) respectively. Thus

$$\mathcal{E}^{\frac{1}{2}}(t) \le \mathcal{E}^{\frac{1}{2}}(0) + \int_0^t e^{(t-t')C_0K}\mathcal{F}(t')dt'$$

and the estimate follows using again the Lemma 7.1.6.

Since all the terms of the estimates are continuous for the norm of $u \in L^2([0,T]; H^1(\mathbb{R}^d))$ with $\partial_t u \in L^1([0,T]; L^2(\mathbb{R}^d))$, by density, this implies that the estimate is true for u in this space. $\square$

Proof of Proposition 7.1.8.

The para-linearization Lemma 7.1.5 and the estimate (7.1.28) immediately give an estimate similar to (7.1.25) with an additional term

$$\gamma M_1(A)\int_0^t e^{K(t-t')}\big\|u(t')\big\|_{L^2} dt' \tag{7.1.29}$$

in the right hand side. Gronwall's lemma, implies (7.1.25) with a new constant $K' = K + \gamma M_1(A)$ which has the same form as K. $\square$

7.1.5 Weak= Strong and uniqueness

For $u \in L^2([0,T]\times\mathbb{R}^d)$, $\partial_t u$ is well defined in $\mathscr{D}'(]0,T[\times\mathbb{R}^d)$ and $A_j\partial_j u \in L^2([0,T];H^{-1}(\mathbb{R}^d)) \subset \mathscr{D}'(]0,T[\times\mathbb{R}^d)$ since the product

$$(a,v) \;\mapsto\; av$$

is well defined from $W^{1,\infty}(\mathbb{R}^d)\times H^{-1}(\mathbb{R}^d) \mapsto H^{-1}(\mathbb{R}^d)$. Therefore, for $f \in \mathscr{D}'(]0,T[\times\mathbb{R}^d)$ the equation

$$Lu := \partial_t u + A(t,x,\partial_x)u = f \tag{7.1.30}$$

makes sense. Such a $u \in L^2([0,T]\times\mathbb{R}^d)$ is called *a weak solution* of the equation.

Lemma 7.1.10. *If $f \in L^1([0,T];L^2(\mathbb{R}^d)$ and $u \in L^2([0,T]\times\mathbb{R}^d)$ satisfy* (7.1.30)*, then*

$$u \in C^0([0,T];H^{-\frac{1}{2}}(\mathbb{R}^d)). \tag{7.1.31}$$

Proof. The function $v = u - \int_0^t f(t')dt'$ belongs to $L^2([0,T],L^2) + C^0([0,T];L^2) \subset L^2([0,T],L^2)$ and $\partial_t v \in L^2([0,T];H^{-1})$. Thus v and hence u belong to $C^0([0,T];H^{-\frac{1}{2}}(\mathbb{R}^d))$. □

In particular for a weak solution $u \in L^2([0,T]\times\mathbb{R}^d)$ of (7.1.30) with $f \in L^1([0,T];L^2(\mathbb{R}^d)$, the trace

$$u_{|t=0} \in H^{-\frac{1}{2}}(\mathbb{R}^d)$$

is well defined and the initial Cauchy condition $u_{|t=0} = h$ makes sense in $\mathscr{D}'(\mathbb{R}^d)$.

Theorem 7.1.11. *Suppose that $u \in L^2([0,T]\times(\mathbb{R}^d)$ satisfies the equation* (7.1.1) *with $f \in L^1([0,T];L^2(\mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$. Then $u \in C^0([0,T];L^2(\mathbb{R}^d))$ and u satisfies the energy estimates* (7.1.25)

Corollary 7.1.12. *If $u \in L^2([0,T]\times(\mathbb{R}^d))$ satisfies* (7.1.1) *with $f = 0$ and $h = 0$, then $u = 0$.*

Let $J_\varepsilon = (1-\varepsilon\Delta)^{-\frac{1}{2}}$. It is the Fourier multiplier with symbol $\jmath_\varepsilon = (1+\varepsilon|\xi|^2)^{-\frac{1}{2}}$. We use the following facts:

- for fixed $\varepsilon > 0$, $\jmath_\varepsilon \in S^{-1}_{1,0}$ and J_ε is of order -1;
- the family $\{\jmath_\varepsilon; \varepsilon \in]0,1]\}$ is bounded in $S^0_{1,0}$ and in particular, the J_ε are uniformly bounded in L^2 and H^s;
- for all $v \in H^s$, $J_\varepsilon v \to v$ in H^s as $\varepsilon \to 0$.

Lemma 7.1.13. *If $u \in L^2([0,T]\times(\mathbb{R}^d)$ satisfies* (7.1.1) *with $f \in L^1([0,T];L^2(\mathbb{R}^d)$, then*

$$LJ_\varepsilon u = f_\varepsilon \in L^1([0,T];L^2(\mathbb{R}^d) \tag{7.1.32}$$

and $f_\varepsilon \to f$ in $f \in L^1([0,T];L^2(\mathbb{R}^d)$.

Proof. By Lemma 7.1.5, the operators $R(t) = A(t,x,\partial_x) - T_{iA(t)}$ satisfies

$$\|R(t)v\|_{L^2} \leq K\|v\|_{L^2}.$$

Thus, $[R(t), J_\varepsilon] = R(t)J_\varepsilon - J_\varepsilon R(t)$ are uniformly bounded in L^2.

The symbolic calculus implies that the commutators $[T_{iA(t)}, J_\varepsilon] = [T_{iA}, T_{\jmath_\varepsilon}]$ are uniformly or order 0 thus there is K such that for all t:

$$\|[T_{iA}, J_\varepsilon]v\|_{L^2} \leq K\|v\|_{L^2}.$$

Adding up, we see that there is K such that for all t and all $\varepsilon \in]0,1]$:

$$\|[A(t,x,\partial_x), J_\varepsilon]v\|_{L^2} \leq K\|v\|_{L^2}. \tag{7.1.33}$$

Moreover, for all $v \in H^1([0,T]\times\mathbb{R}^d)$ and all t, the commutator $[A(t,x,\partial_x), J_\varepsilon]v$ tends to 0 in L^2 since each term $A(t,x,\partial_x)J_\varepsilon v$ and $J_\varepsilon A(t,x,\partial_x)v$ converge to $A(t,x,\partial_x)v$. Using the uniform bound (7.1.33) and the density of H^1 into L^2, this shows that for all $u \in L^2([0,T]\times\mathbb{R}^d)$

$$g_\varepsilon := [A(t,x,\partial_x), J_\varepsilon]u(t) \ \to \ 0 \qquad \text{in } L^2.$$

In particular

$$\int_0^T \|g_\varepsilon(t)\|_{L^2} dt \ \to \ 0 \qquad \text{as } \varepsilon \to 0.$$

Because J_ε and ∂_t commute, there holds (in the sense of distributions)

$$\partial_t J_\varepsilon + A(t,x,\partial_x)J_\varepsilon = J_\varepsilon f + g_\varepsilon \tag{7.1.34}$$

and the lemma follows. □

Proof of Theorem 7.1.11. Because $J_\varepsilon u \in C^0([0,T];H^1)$ and

$$\partial_t J_\varepsilon u = f_\varepsilon - A(t,x,\partial_x)J_\varepsilon u \in L^1([0,T];H^1) + C^0([0,T];L^2)$$

one can apply Theorem 7.1.3 to $J_\varepsilon u - J_{\varepsilon'}u$. The corresponding estimate imply that the $J_\varepsilon u$ form a Cauchy family in $C^0([0,T];L^2)$ as $\varepsilon \to 0$. Thus $J_\varepsilon u$ converge in $C^0([0,T];L^2)$. Since $J_\varepsilon u \to u$ in $L^2([0,T]\times\mathbb{R}^d)$, this shows that $u \in C^0([0,T];L^2)$ and $J_\varepsilon u \to u$ in $C^0([0,T];L^2)$.

Theorem 7.1.3 can also be applied to J_ε and passing to the limit in the estimates satisfied by $J_\varepsilon u$ proves that u satisfies (7.1.4). □

7.1.6 Existence

To finish the proof of Theorem 7.1.3, it remains to prove the existence of weak solutions.

Theorem 7.1.14. *For $f \in L^1([0,T]; L^2(\mathbb{R}^d))$ and $h \in L^2(\mathbb{R}^d))$ the Cauchy problem* (7.1.1) *has a unique solution $u \in L^2([0,T] \times \mathbb{R}^d)$ which therefore belongs to $C^0([0,T]; L^2(\mathbb{R}^d))$ and satisfies the energy estimate* (7.1.25).

Proof. Consider the equation

$$\partial_t u_\varepsilon + A(t,x,\partial_x) J_\varepsilon u_\varepsilon = f, \qquad u_{\varepsilon|t=0} = h. \tag{7.1.35}$$

For each fixed $\varepsilon > 0$, $A(t,x,\partial_x)J_\varepsilon$ is bounded in L^2 and the theorem of Cauchy-Lipschitz implies that there is a solution $u \in C^0([0,T]; L^2)$.

The main point is that all the estimates proved for $\partial_t + A(t,x,\partial_x)$ are satisfied for $\partial_t + A(t,x,\partial_x)J_\varepsilon$, uniformly in ε. Indeed:

- Lemma 7.1.5 implies that the errors $A(t,x,\partial_x)u - T_{iA\jmath_\varepsilon}$ are uniformly bounded in L^2,

- the proof of Proposition 7.1.9 applies to $\partial_t + T_{iA\jmath_\varepsilon}$ because $S(t,x,\xi)$ symmetrizes $A(t,x,\xi)\jmath_\varepsilon(\xi)$ and provides us with uniform estimates since the family of symbols $A(t,x,\xi)\jmath_\varepsilon(\xi)$ is bounded in Γ^1_1.

The uniform estimates for $\partial_t + A(t,x,\partial_x)J_\varepsilon$ imply that the sequence u_ε is bounded in $C^0([0,T]; L^2(\mathbb{R}^d))$. Using the equation, we see that $\partial_t u_\varepsilon - f$ is bounded in $C^0([0,T]; H^{-1}(\mathbb{R}^d))$.

Therefore, by Ascoli-Arzela theorem, there is a subsequence, still denoted bu u_ε, which converges in $C^0([0,T]; L^2_w(\mathbb{R}^d))$, where L^2_w denotes the space L^2 equipped with the weak topology. There is no difficulty to pass to the limit in the equation, and the limit $u \in C^0([0,T]; L^2_w(\mathbb{R}^d)) \subset L^2([0,T] \times \mathbb{R}^d)$ is a (weak) solution of the Cauchy problem. ☐

7.2 The H^s linear theory

7.2.1 Statement of the result

We always assume in this section that Assumption 7.1.2 is satified. We now assume that $s > \frac{d}{2} + 1$ is given and that Assumption 7.1.1 is strengthened as follows:

Assumption 7.2.1. *The matrices A_j have coefficients in $W^{1,\infty}$ and for all $k \in \{1, \dots, d\}$, $\partial_{x_k} A_j \in L^\infty([0,T]; H^{s-1}(\mathbb{R}^d))$.*

For example, the coefficients can be of the form *constant* + function in $C^0(H^s) \cap C^1(H^{s-1})$.

Theorem 7.2.2. *For $f \in L^1([0,T]; H^s(\mathbb{R}^d))$ and $h \in H^s(\mathbb{R}^d))$ the Cauchy problem (7.1.1) has a unique solution $u \in C^0([0,T]; H^s(\mathbb{R}^d))$. Moreover, here are constants C and K_s such that for all f and h the solution u satisfies:*

$$\|u(t)\|_{H^s} \le Ce^{K_s t}\|u(0)\|_{H^s} + C\int_0^t Ce^{K_s(t-t')}\|Lu(t')\|_{H^s} dt'. \tag{7.2.1}$$

The constant C is still of the form (7.1.26). The form of the constant K_s is given in (7.2.6) below.

7.2.2 Paralinearisation

Notations. Parallel to (7.1.6), introduce

$$M_{H^s}(A) = \sum_j \sup_{t\in[0,T]} \|\nabla_x A_j(t,\cdot)\|_{H^{s-1}(\mathbb{R}^d)}. \tag{7.2.2}$$

Proposition 5.2.2 implies that

$$\|A_j\partial_{x_j}u(t) - T_{iA_j\xi_j}u(t)\|_{H^s} \le \gamma\|\nabla_x A_j(t,\,\cdot\,)\|_{H^{s-1}}\ \|\nabla_x u(t)\|_{L^\infty}. \tag{7.2.3}$$

Therefore:

Lemma 7.2.3. *There is a constant γ such that for $u \in C^0([0,T]; H^1)$:*

$$\|A(t,x,\partial_x)u(t) - T_{iA}u(t)\|_{H^s} \le \gamma M_{H^s}(A)\|u(t)\|_{H^s}. \tag{7.2.4}$$

7.2.3 Estimates

Proposition 7.2.4. *There are constants C and K of the form (7.1.26) and (7.1.27) such that for all $u \in C^1([0,T]; H^s) \cap C^0([0,T]; H^{s+1})$ there holds*

$$\|u(t)\|_{H^s} \le Ce^{Kt}\|u(0)\|_{H^s} + C\int_0^t e^{K(t-t')}\|\partial_t + T_{iA}u(t')\|_{H^s} dt'. \tag{7.2.5}$$

Proof. Let $v = (1-\Delta_x)^{\frac{1}{2}s}u = T_{(1+|\xi|^2)^{\frac{1}{2}s}}u$. By the symbolic calculus,

$$\partial_t v + T_{iA}v = (1-\Delta_x)^{\frac{1}{2}s} f + P_s u$$

with P of order s and of norm $O(M_1(A))$ from H^s to L^2. Thus the L^2 estimate for v implies an H^s estimate (7.2.5) with the additional term

$$CM_1(A)\int_0^t e^{K(t-t')}\|u(t')\|_{H^s}dt'.$$

in the right hand side. This implies (7.2.5) with a new constant $K' = K + CM_1(A)$ which is still of the form (7.1.27). □

Using this estimate together with (7.2.4) and Gronwall's lemma immediately implies that following:

Proposition 7.2.5. *There are constants C of the form* (7.1.26) *and K_s of the form*

$$K_s = K' + C'M_{H^s}(A) \tag{7.2.6}$$

with C' and K' of the form (7.1.26) *and* (7.1.27) *respectively, such that the energy estimate* (7.2.1) *is satisfied for all $u \in C^1([0,T];H^s) \cap C^0([0,T];H^{s+1})$*

Remark 7.2.6. Ot is noticeable that for the *para-differential equation*, the estimate depends only the Lipschitz norm of the coefficients.

7.2.4 Smoothing effect in time

Using the mollifiers J_ε as in the proof of Theorem 7.1.11, one obtains the following similar result.

Proposition 7.2.7. *Suppose that $u \in L^2([0,T];H^s(\mathbb{R}^d))$ is a solution of the Cauchy problem* (7.1.1) *with $f \in L^1([0,T];H^s(\mathbb{R}^d))$ and $h \in H^s(\mathbb{R}^d)$. Then $u \in C^0([0,T];H^s(\mathbb{R}^d))$ and u satisfies the energy estimates* (7.2.1).

7.2.5 Existence

Theorem 7.2.8. *For $f \in L^1([0,T];H^s\mathbb{R}^d))$ and $h \in H^s(\mathbb{R}^d))$ the Cauchy problem* (7.1.1) *has a unique solution $u \in L^2([0,T];H^s(\mathbb{R}^d))$ which therefore belongs to $C^0([0,T];H^s(\mathbb{R}^d))$ and satisfies the energy estimate* (7.2.1).

Proof. The proof is similar to the proof of Theorem 7.1.3. We consider the mollified equation

$$\partial_t u_\varepsilon + A(t,x,\partial_x J_\varepsilon)u_\varepsilon = f, \qquad u_{\varepsilon|t=0} = h. \tag{7.2.7}$$

For each fixed $\varepsilon > 0$, $A(t,x,\partial_x)J_\varepsilon$ is bounded in H^s and the theorem of Cauchy-Lipschitz implies that there is a solution $u \in C^0([0,T];H^s)$.

Again, we use that the estimates are uniform in ε, so that the sequence u_ε is bounded in $C^0([0,T];H^s(\mathbb{R}^d))$ and $\partial_t u_\varepsilon - f$ is bounded in $C^0([0,T];H^{s-1}(\mathbb{R}^d))$. Therefore, a subsequence converges in $C^0([0,T];H^s_w(\mathbb{R}^d))$, where H^s_w denotes the space H^s equipped with the weak topology. The limit $u \in L^2([0,T];H^s(\mathbb{R}^d))$ is a (weak) solution of the Cauchy problem.

We conclude, using Proposition 7.2.7. □

7.3 Quasi-linear systems

7.3.1 Statement of the results

We consider a first order $N \times N$ quasi-linear system

$$\text{(7.3.1)} \qquad \begin{cases} \partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u = f + F(u), \\ u_{|t=0} = h. \end{cases}$$

Assumption 7.3.1. *The matrices A_j are C^∞ functions of $u \in \mathbb{R}^N$. F is a smooth function of u and $F(0) = 0$.*

For simplicity, we assume here that the coefficients A_j do not depend on the variables (t,x). The extension to systems with coefficients $A_j(t,x,u)$ is left as an exercise.

The symbol of the equation is

$$\text{(7.3.2)} \qquad A(u,\xi) = \sum_{j=1}^{d} \xi_j A_j(u)$$

Assumption 7.3.2 (Hyperbolicity)**.** *There is a $N \times N$ matrix $S(u,\xi)$, homogeneous of degree 0 in ξ, with entries C^∞ in (u,ξ) when $\xi \neq 0$ and such that:*

i) $S(u,\xi)$ is self adjoint and definite positive,
ii) For all (u,ξ), $S(u\xi)A(t,x,\xi)$ is self-adjoint.

We consider a Sobolev index $s > \frac{d}{2} + 1$ which is fixed throughout this section.

Theorem 7.3.3. *For $f \in C^0([0,T];H^s(\mathbb{R}^d)$ and $h \in H^s(\mathbb{R}^d)$, there is $T' > 0$ and a unique solution $u \in C^0([0,T'];H^s(\mathbb{R}^d))$ of the Cauchy problem* (7.3.1)*.*

An estimate from below of T' is given in the proof of the theorem.

Uniqueness allows to define the maximal time of existence :

T^* is the supremum of $T' \in [0,T]$ such that the Cauchy problem has a solution $u \in C^0([0,T']; H^s(\mathbb{R}^d))$.

The theorem implies that $T^* > 0$. By uniqueness, the solution u is therefore defined for all $t < T^*$ and $u \in C^0([0,T^*[; H^s(\mathbb{R}^d))$.

Theorem 7.3.4. *Either $T^* = T$ or*

$$\limsup_{t \to T^*} \|u\|_{L^\infty([0,t]\times\mathbb{R}^d)} + \|\nabla_{t,x} u\|_{L^\infty([0,t]\times\mathbb{R}^d)} = +\infty. \tag{7.3.3}$$

7.3.2 Local in time existence

We consider the iterative scheme defined by $u_0(t,x) = h(x)$ and for $n \geq 0$:

$$\begin{cases} \partial_t u_{n+1} + \sum_{j=1}^d A_j(u_n)\partial_j u_{n+1} = f + F(u_n), \\ u_{n+1|t=0} = h. \end{cases} \tag{7.3.4}$$

Lemma 7.3.5. *The u_n are defined for all n and*

$$u_n \in C^0([0,T]; H^s(\mathbb{R}^d)), \qquad \partial_t u_n \in C^0([0,T]; H^{s-1}(\mathbb{R}^d)). \tag{7.3.5}$$

Proof. This is true for u_0. Suppose that u_n satisfies (7.3.5). Therefore, u_n belongs to $W^{1,\infty}$ as well as the coefficients $A_j(u_n(t,x))$. Moreover, applying Theorem 5.2.6 to $A_j(u) - A_j(0)$, implies that $\nabla_x A_j(u_n) \in C^0([0,T]; H^{s-1})$. Moreover, the linear equation (7.3.4) admitd the symmetrizers $S(u_n(t,x),\xi)$ which satisfy the conditions of Assumption 7.1.2. Therefore Theorem 7.2.2 can be applied, and (7.3.5) has a unique solution $u_{n+1} \in C^0([0,T]; H^s)$. The property $\partial_t u_{n+1} \in C^0([0,T]; H^{s-1}(\mathbb{R}^d))$ follows from the equation. $\square$

Lemma 7.3.6. *There is $T' \in]0,T]$, such that the sequences u_n and $\partial_t u_n$ are bounded in $C^0([0,T']; H^s(\mathbb{R}^d))$ and in $C^0([0,T']; H^{s-1}(\mathbb{R}^d))$ respectively.*

Proof. We prove by induction that there are $T' > 0$ and constants m, R and R_1 such that

$$\|u_n\|_{L^\infty([0,T']\times\mathbb{R}^d)} \leq m, \tag{7.3.6}$$

$$\sup_{t\in[0,T']} \|u_n(t)\|_{H^s(\mathbb{R}^d)} \leq R, \tag{7.3.7}$$

$$\sup_{t\in[0,T']} \|\partial_t u_n(t)\|_{H^{s-1}(\mathbb{R}^d)} \leq R_1. \tag{7.3.8}$$

We use the energy estimates (7.2.1) on $[0, T']$ for the linear problem (7.3.4), assuming u_n satisfies the estimates above. There are constants C_n and K_n, depending on u_n, such that

$$\big\|u_{n+1}(t)\big\|_{H^s} \le C_n e^{K_n t}\big\|h\big\|_{H^s} + C_n \int_0^t e^{K_n(t-t')}\big\|f(t') + F(u_n(t'))\big\|_{H^s} dt'.$$

The symmetrizer is $S_n(t,x,\xi) = S(u_n(t,x),\xi)$. If u_n satisfies the induction hypothesis, Assumption 7.3.2 implies that the semi norms occurring in the definition of the the constants satisfy

$$M_0(S_n^{\pm};k) \le F(m), \qquad M_1(S_n^{\pm};k) \le F_1(R)$$

where F and F_1 are increasing functions of their argument. Therefore, there are functions $C(m)$ and $K(m,R,R_1)$ such that the constants C_n and K_n satisfy $C_n \le C(m)$ and $K_n \le K(m,R,R_1)$.

The H^s norm of $F(u_n)$ is also estimated by $K(R)$, and finally, we see that there are constants $C = C(m)$ and $K = K(m,R,R_1)$ such that

$$\big\|u_{n+1}(t)\big\|_{H^s} \le Ce^{Kt}\big\|h\big\|_{H^s} + tCe^{Kt}(\Phi + K) \tag{7.3.9}$$

where

$$\Phi = \sup_{t\in[0,T]} \big\|f(t)\big\|_{H^s(\mathbb{R}^d)}$$

Using the equation, this implies that there is a function $D(R)$

$$\big\|\partial_t u_{n+1}(t)\big\|_{H^{s-1}} \le \Phi + D(R)\big\|u_{n+1}(t)\big\|_{H^s}. \tag{7.3.10}$$

We first choose

$$m > \big\|h\big\|_{L^\infty(\mathbb{R}^d)} \tag{7.3.11}$$

Next we choose

$$R > C(m)\big\|h\big\|_{H^s(\mathbb{R}^d)}, \tag{7.3.12}$$

$$R_1 > \Phi + RD(R). \tag{7.3.13}$$

Therefore, if T' is small enough,

$$C(m)e^{T'K}\big(\big\|h\big\|_{H^s(\mathbb{R}^d)} + T'(\Phi + K)\big) \le R, \tag{7.3.14}$$

and the energy estimates (7.3.9) and (7.3.10) imply that u_{n+1} satisfies (7.3.7) and (7.3.8)

The estimate of $\partial_t u_{n+1}$ implies that

$$\big\|u_{n+1}(t) - h\big\|_{L^\infty} \le \gamma \big\|u_{n+1}(t) - h\big\|_{H^{s-1}} \le tR_1, \tag{7.3.15}$$

and thus,

$$\big\|u_{n+1}(t)\big\|_{L^\infty} \le \big\|h\big\|_{L^\infty} + tR_1.$$

If T' is small enough, the right hand side is $\le m$, implying that u_{n+1} satisfies (7.3.6). □

Lemma 7.3.7. *The sequence u_n is a Cauchy sequence in $C^0([0,T'];L^2(\mathbb{R}^d))$.*

Proof. Set $v_n := u_{n+1} - u_n$. For $n \ge 1$, it satisfies

$$\begin{cases} \partial_t v_n + \displaystyle\sum_{j=1}^d A_j(u_n)\partial_j v_n = g_n, \\ v_{n+1|t=0} = 0 \end{cases} \tag{7.3.16}$$

with

$$g_n = F(u_n) - F(u_{n-1}) + \sum_{j=1}^d \big(A_j(u_{n-1}) - A_j(u_n)\big)\partial_j u_n.$$

Using the uniform bounds for u_n and u_{n+1}, we see that there is a constant R such that for all n and all $(t,x) \in [0,T'] \times \mathbb{R}^d$:

$$|g_n(t,x)| \le R|v_{n-1}(t,x)|$$

The uniform bounds also imply that the L^2 energy estimates for (7.3.16) are satisfied with constants independent of n. Therefore, there is a M, independent of n, such that for all $n \ge 1$:

$$\big\|v_n(t)\big\|_{L^2} \le R \int_0^t \big\|v_{n-1}(t')\big\|_{L^2} dt'.$$

Hence

$$\big\|v_n(t)\big\|_{L^2} \le \frac{t^n R^n}{n!} \sup_{t' \in [0,T']} \big\|v_0(t')\big\|_{L^2}$$

Thus the series $\sum v_n$ converges in $C^0([0,T'];L^2(\mathbb{R}^d))$. □

Proof. Proof of Theorem 7.3.3 The uniform bounds and the convergence in $C^0([0,T'];L^2(\mathbb{R}^d))$ imply that the sequence u_n converges in $C^0([0,T'];H^{s'}(\mathbb{R}^d))$ for all $s'<s$. Similarly, $\partial_t u_n$ converges in $C^0([0,T'];H^{s'-1}(\mathbb{R}^d))$ Choosing $s'>\frac{d}{2}+1$, this implies a uniform convergence in C^0 of u_n and $\nabla_{t,x}u_n$. Thus the limit u is solution of the equation. Moreover, $u\in C^0([0,T'];H^{s'}(\mathbb{R}^d))$ and $u\in L^\infty([0,T];H^s)$ and $\partial_t u\in L^\infty([0,T];H^{s-1})$.

The next step consists in considering (7.3.1) as a linear equation in $u\in L^2([0,T'];H^s)$ with coefficients A_j in $L^\infty([0,T'],H^s)$. Proposition 7.2.7 implies that $u\in C^0([0,T];H^s)$. □

Remark 7.3.8. The proof above shows that the time of existence is uniformly estimated from below by a uniform T' when the data f and h remain in bounded sets.

7.3.3 Blow up criterion

The proof of Theorem 7.3.4 is based on the following a priori estimate:

Theorem 7.3.9. *For all M, there are constants $C(M)$ and $K(M)$ such that for all solution $u\in C^1([0,T];H^s)\cap C^0([0,T];H^{s+1})$ which satisfies*

$$\|u\|_{W^{1,\infty}([0,T]\times\mathbb{R}^d)}\le M \tag{7.3.17}$$

there holds

$$\|u(t)\|_{H^s}\le Ce^{Kt}\|u(t)\|_{H^s}+C\int_0^t Ce^{K(t-s)}\|f(s)\|_{H^s}ds. \tag{7.3.18}$$

Proof. With $A_j=A_j(u)$ the paralinearisation theorem implies that

$$\|A(u,\partial_x)u(t)-T_{iA}u(t)\|_{H^s}\le K\|u(t)\|_{H^s}. \tag{7.3.19}$$

with $K=K(M)$. Then, the estimates follows from Proposition 7.2.4 for the para-differential equation and Gronwall's Lemma to absorb the integral in $\|u(t')\|_{H^s}$ which appears in the right hand side. □

Proof of Theorem 7.3.3.

Suppose that $T^*<T$ but that there is M such that for all $t<T^*$

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)}+\|\nabla_{t,x}u(t)\|_{L^\infty(\mathbb{R}^d)}.\le M$$

By Theorem 7.3.9, the H^s norm of $u(t)$ remains bounded by a constant R.

Following the Remark 7.3.8, there is $T' > 0$ such that the Cauchy problem (7.3.1) with f in a ball of $C^0([0,T];H^s)$ and $\|h\|_{H^s} \le R$ has a solution in $C^0([0,T'];H^s)$. We apply this result for the Cauchy problem with initial time $T_1 = T^* - T'/2$ and initial data $u(T_1)$ which satisfies $\|u(T_1)\|_{H^s} \le R$. Therefore this initial value problem has a solution in $C^0([T_1,T_2];H^s)$ with $T_2 = \min(T_1 + T', T)$. By uniqueness, it must be equal to u on $[T_1, T^*[$, and therefore u has an extension $\tilde{u} \in C^0([0,T_2];H^s)$ solution of the Cauchy problem. Since $T_2 > T^*$, this contradicts the definition of T^* and the proof of the theorem is complete. □

Chapter 8

Systems of Schrödinger equations

In this chapter, we give another application of the symbolic calculus to the analysis of the Cauchy problem for systems of Schrödinger equations. The Schrödinger equation is very classical in optics as it models the propagation of a coherent beam along long distances. The dispersive character of Schröndinger equation encounters for the dispersion of light in the directions transverse to the beam. The coupling of such equations thus models the interaction of several beams. For an introduction to nonlinear optics we can refer for instance to to [Blo, Boy, NM] and for examples of coupled Schrödinger equations, to [CC, CCM].

8.1 Introduction

Motivated by nonlinear optics, we consider systems of scalar Schrödinger equations:

$$\partial_t u_j + i\lambda_j \Delta_x u_j = \sum_{k=1}^{N} b_{j,k}(u, \partial_x) u_k, \qquad j \in \{1, \dots, N\}, \tag{8.1.1}$$

where the λ_j are real and the $b_{j,k}(u, \partial_x)$ are first order in ∂_x. See [CC] and the references therein. In general, the nonlinear terms depend on u and $\overline{u}$:

$$\partial_t u_j + i\lambda_j \Delta_x u_j = \sum_{k=1}^{N} b_{j,k}(u, \partial_x) u_k + c_{j,k}(u, \partial_x) \overline{u}_k \tag{8.1.2}$$

where the $c_{j,k}(u, \partial_x)$ are also of first order in ∂_x. Introducing u and $\overline{u}$ as separate unknowns reduces to the form (8.1.1) for a doubled system.

This system can be written in a more condensed form

$$\partial_t u + iA(\partial_x)u + B(t, x, u, \partial_x)u = 0 \tag{8.1.3}$$

with $A = \text{diag}(\lambda_j)$ second order and B a first order $N \times N$ system

$$B(t, x, u, \partial_x) = \sum_{j=1}^{d} B_j(t, x, u)\partial_{x_j}. \tag{8.1.4}$$

For the local existence of smooth solutions, the easy case is when the first order part, $B(u, \partial_x)$ in the right hand side is hyperbolic symmetric, that is when the matrices B_j are self adjoint. In this case, there are obvious L^2 estimates (for the linearized equations) followed by H^s estimates obtained by differentiating the equations as in Section 3.3. They imply the local well-posedness of the Cauchy problem for (8.1.1) in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$.

But in many examples, $B(u, \partial_x)$ *is not symmetric* and even more $\partial_t - B(u, \partial_x)$ is not hyperbolic, implying that the Cauchy problem for $\partial_t u - B(u, \partial_x)u = 0$ is ill posed. However, the Cauchy problem for (8.1.1) may be well posed even if it is ill posed for the first order part. The main objective of this chapter is to show that under suitable assumptions, *one can use the symbolic calculus to transform nonsymmetric systems* (8.1.1) *into symmetric ones.*

8.1.1 Decoupling

As an example, consider the Cauchy problem for

$$\begin{cases} \partial_t u + i\Delta_x u + \partial_{x_1} v = 0, \\ \partial_t v - i\Delta_x v - \partial_{x_1} u = 0. \end{cases} \tag{8.1.5}$$

On the Fourier side, it reads $\partial_t \hat{U} + iA(\xi)\hat{U} = 0$ with

$$A(\xi) = \begin{pmatrix} -|\xi|^2 & \xi_1 \\ -\xi_1 & +|\xi|^2 \end{pmatrix}$$

The eigenvalues of A are not real for all ξ, but their imaginary parts are uniformly bounded, implying that the Cauchy problem for (8.1.5) is well posed in H^s.

More generally, when B has constant coefficients, the Fourier analysis leads to consider the matrix

$$-\mathrm{diag}(\lambda_j|\xi|^2) + \big(b_{j,k}(\xi)\big)$$

When the λ_j are pairwise distinct, an elementary perturbation analysis shows that the eigenvalues of this matrix are

$$-\lambda_j|\xi|^2 + b_{j,j}(\xi) + O(1).$$

Therefore, their imaginary parts are uniformly bounded if $\mathrm{Im}\, b_{j,j}(\xi) = 0$ for all j, in which case the Cauchy problem is well posed in H^s.

This analysis can be extended to variable coefficient systems and next to nonlinear systems, using the symbolic calculus developed in Chapter 6. For instance, we will prove the following result:

Theorem 8.1.1. *If the λ_j are real and pairwise distinct and if the diagonal terms $b_{j,j}(u,\partial_x)$ have real coeficients, then the Cauchy problem for* (8.1.1) *is well posed in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$.*

Analogously, for systems (8.1.2), we prove the following result:

Theorem 8.1.2. *Suppose that*
- *the λ_j are real and pairwise distinct*
- *the diagonal terms $b_{j,j}(u,\partial_x)$ have real coeficients,*
- *$c_{j,k}(u,\partial_x) = c_{k,j}(u,\partial_x)$ for all pair (j,k) such that $\lambda_j + \lambda_k = 0$.*

Then the Cauchy problem for (8.1.2) *is well posed in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$.*

8.1.2 Further reduction

In the scalar case, the lack of symmetry of the first order term is a real problem: for instance it has been noticed for a long time that the Cauchy problem for $\partial_t - i\Delta_x + i\partial_{x_1}$ is ill posed in H^∞. A more precise result has been given by S.Mizohata ([Miz]): for $\partial_t - i\Delta_x + b(x)\cdot\nabla_x$ a necessary condition for the well posedness of the Cauchy problem in H^s is that

$$\int_{\mathbb{R}} |\omega \cdot \mathrm{Im}\, b(x + s\omega)|ds \le C \tag{8.1.6}$$

for all $x \in \mathbb{R}^d$ and $\omega \in S^{d-1}$. Moreover, a sufficient condition is that b and its derivative satisfy (8.1.6). This result is also a consequence of a symbolic calculus as shown in [KPV] for instance, but the details are out of the scope of these elementary notes.

The idea of using pseudo-differential symmetrizers is not very far from the proof used in [CC] where the symmetry is obtained after differentiation of the equations and clever linear recombination: this amounts to use differential symmetrizers. The idea of using pseudo-differential operators to reduce oneself to the symmetric case has been used in the literature for a long time (see e.g. [Ch] and [KPV] and the references therein).

One property which is hidden behind these analyses is *dispersive character* of Schrödinger equations. We stress that the results presented in this chapter *do not* give the full strength of the dispersive properties, and in particular do not mention (no use) the *local smoothing properties.* Our goal is to use the easy symbolic calculus to reduce the analysis of systems to the analysis of scalar equations, where the specific known results can be applied.

8.2 Energy estimates for linear systems

8.2.1 The results

We consider a slightly more general framework and $N \times N$ systems

$$Lu := \partial_t u + iA(\partial_x)u + B(t, x, \partial_x)u = f \tag{8.2.1}$$

with A second order and B first order :

$$A(\partial_x) = \textstyle\sum_{j,k=1}^{d} A_{j,k}\partial_{x_j}\partial_{x_k}, \tag{8.2.2}$$

$$B(t, x, \partial_x) = \textstyle\sum_{j=1}^{d} B_j(t, x)\partial_{x_j}. \tag{8.2.3}$$

With the example (8.1.1) in mind, we assume that A is smoothly block-diagonalizable:

Assumption 8.2.1. *For all $\xi \in \mathbb{R}^n \backslash \{0\}$, $A(\xi) = \sum A_{j,k}\xi_j\xi_k$ is self-adjoint with eigenvalues of constant multiplicity.*

This implies that there are smooth real eigenvalues $\lambda_p(\xi)$ and smooth self-adjoint eigenprojectors $\Pi_p(\xi)$ such that

$$A(\xi) = \sum_p \lambda_p(\xi)\Pi_p(\xi). \tag{8.2.4}$$

In particular, it means that the system $\partial_t + iA(\partial_x)$ can be diagonalized using Fourier mutlipliers.

The smoothness of the coefficients B_j with respect to x is measured in spaces $W^{k,\infty}(\mathbb{R}^d)$. When $k \geq 0$, these spaces are defined in Chapter 4. We will also use the space $W^{-1,\infty}(\mathbb{R}^d)$ of distributions $u = u_0 + \sum \partial_{x_j} u_j$ with $u_j \in L^\infty(\mathbb{R}^d)$. In applications to nonlinear problems, these conditions will appear naturally through the Sobolev injection $H^{s-1}(\mathbb{R}^d) \subset W^{-1,\infty}(\mathbb{R}^d)$ when $s > \frac{d}{2}$.

We denote by $B(t,x,\xi) := \sum \xi_j B_j(t,x)$ the symbol of $\frac{1}{i}B(t,x,\partial_x)$. The diagonalization of $A(\xi)$ leads to consider the blocks $\Pi_p(\xi)B(t,x,\xi)\Pi_q(\xi)$.

We first prove energy estimates under the following assumptions for B.

Assumption 8.2.2. *i)* [Symmetry of the diagonal blocks] *For all p, t and x, the matrix $\Pi_p(\xi)B(t,x,\xi)\Pi_p(\xi)$ is self adjoint.*

ii) [Smoothness] *The matrices $B_j(t,x)$ belong to $C^0([0,T];W^{1,\infty}(\mathbb{R}^d)$ and $\partial_t B_j(t,x)$ belong to $C^0([0,T];W^{-1,\infty}(\mathbb{R}^d)$.*

Remark 8.2.3. There is no assumption on the spectrum of $B(t,x,u,\xi)$. Only the diagonal blocks $\Pi_p(\xi)B(t,x,\xi)\Pi_p(\xi)$ intervene in i).

The smoothness of the coefficients B_j obeys the rule 1-time derivative = 2-space derivatives, which is natural from the equations.

Theorem 8.2.4. *Under Assumptions 8.2.1 and 8.2.2, all $u \in C^1([0,T];H^2(\mathbb{R}^d)$ satisfies the energy estmate*

$$\big\|u(t)\big\|_{L^2} \leq C_0(K_0)e^{tC_1(K_1)}\Big(\big\|u(0)\big\|_{L^2} + \int_0^t \big\|Lu(t')\big\|_{L^2}dt'\Big) \tag{8.2.5}$$

where the constants C_0 and C_1 depend only on K_0 and K_1 respectively with

$$K_0 = \sup_j \|B_j\|_{L^\infty([0,T]\times\mathbb{R}^d)}, \tag{8.2.6}$$

$$K_1 = \sup_j \|B_j\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))} + \|\partial_t B_j\|_{L^\infty([0,T];W^{-1,\infty}(\mathbb{R}^d))}. \tag{8.2.7}$$

8.2.2 Proof of Theorem 8.2.4

We use the paradifferential calculus and the notations introduced in the previous chapters. In particular, following Definition 7.1.4, $\Gamma_k^m([0,T]\times\mathbb{R}^d)$ denotes the space of symbols $a(t,x,\xi)$ which are smooth and homogeneous of degree m in ξ and such that for all $\alpha \in \mathbb{N}^d$;

$$\sup_{t\in[0,T]}\sup_{|\xi|=1} \big\|\partial_\xi^\alpha a(t,\cdot,\xi)\big\|_{W^{k,\infty}(\mathbb{R}^d)} < +\infty \tag{8.2.8}$$

This definition immediately extends to the case $k = -1$.

Suppose that

$$B_j \in C^0([0,T], W^{1,\infty}(\mathbb{R}^d)). \tag{8.2.9}$$

The paralinearization Theorem 5.2.9 implies that $f_1 := B(t,x,\partial_x)u - T_{iB}u$ satisfies

$$\|f_1(t)\|_{L^2} \leq \gamma K_1 \|u(t)\|_{L^2}. \tag{8.2.10}$$

Therefore if u satisfies the equation (8.2.1), it also satisfies the paralinearized equation:

$$\partial_t u + iA(\partial_x)u + T_{iB}u = f + f_1. \tag{8.2.11}$$

Consider first the symmetric case:

Proposition 8.2.5. *There is a constant γ such that for $P \in \Gamma_1^1([0,T]\times\mathbb{R}^d)$ satisfying $P = -P^*$ and $u \in C^1([0,T];H^2(\mathbb{R}^d))$, there holds:*

$$\big\|u(t)\big\|_{L^2} \leq e^{tC}\big\|u(0)\big\|_{L^2} + \int_0^t e^{(t-t')C}\big\|f(t')\big\|_{L^2}dt' \tag{8.2.12}$$

where $f = \partial_t u + iA(\partial_x)u + T_P u$ and

$$C = \gamma M_1^1(P;n), \qquad n = [\frac{d}{2}] + 2.$$

Proof. The operator $A(\partial_x)$ is self adjoint and the symbolic calculus implies that $T_P + (T_P)^*$ is of order 0. Therefore

$$\partial_t\big\|u(t)\big\|_{L^2} \leq 2\mathrm{Re}\,\big(f(t),u(t)\big)_{L^2} + C\big\|u(t)\big\|_{L^2}$$

and the estimate follows. □

Next, consider the case where only the diagonal blocks are symmetric.

Proposition 8.2.6. *There is a constant γ such that for $P \in \Gamma_1^1([0,T]\times\mathbb{R}^d)$ satisfying $\partial_t P \in \Gamma_{-1}^1([0,T]\times\mathbb{R}^d)$ and $\Pi_p \mathrm{Re}\, P \Pi_p = 0$ for all p, and for all $u \in C^1([0,T];H^2(\mathbb{R}^d))$, there holds:*

$$\big\|u(t)\big\|_{L^2} \leq e^{tC}\big\|u(0)\big\|_{L^2} + \int_0^t e^{(t-t')C}\big\|f(t')\big\|_{L^2}dt' \tag{8.2.13}$$

where $f = \partial_t u + iA(\partial_x)u + T_P u$ and

$$C = \gamma\Big(M_1^1(P;n) + M_{-1}^1(\partial_t P;n)\Big), \qquad n = [\frac{d}{2}] + 2 \tag{8.2.14}$$

Proof. Set

$$V(t,x,\xi) = \sum_{p\neq q} \frac{i}{\lambda_p(\xi) - \lambda_q(\xi)} \Pi_p(\xi)(P(t,x,\xi))\Pi_q(\xi)\theta_\lambda(\xi) \tag{8.2.15}$$

where $\theta_\lambda(\xi) = \theta(\lambda^{-1}\xi)$ and $\theta \in C^\infty(\mathbb{R}^d)$ is such that $\theta(\xi) = 0$ for $|\xi| \leq 1$ and that $\theta(\xi) = 1$ for $|\xi| \geq 2$. It satisfies the commutation property:

$$P - [V, iA] = P(1-\theta_\lambda) + Q, \qquad Q := \sum_p \Pi_p P \Pi_p \theta_\lambda \tag{8.2.16}$$

where, by assumption, $\operatorname{Re} Q = 0$.

Moreover, the definition (8.2.15) shows that for $\lambda \geq 1$, V is a symbol of degree -1 and has the same smoothness in (t,s) as B:

$$V \in \Gamma_1^{-1}([0,T]\times\mathbb{R}^d), \qquad \partial_t V \in \Gamma^{-1}1_{-1}([0,T]\times\mathbb{R}^d). \tag{8.2.17}$$

Furthermore, the semi-norms of V are bounded by the corresponding semi-norms of P, uniformly in $\lambda \geq 1$.

Thus,

$$\left\|T_V u\right\|_{L^2} \leq \gamma M_0^1(P;n) \left\|\theta_\lambda(D_x)u\right\|_{H^{-1}} \leq \frac{\gamma M_0^1(P;n)}{\lambda} \left\|\theta_\lambda(D_x)u\right\|_{L^2}.$$

In particular, $\left\|T_V u\right\|_{L^2} \leq \frac{1}{2}\left\|\theta_\lambda(D_x)u\right\|_{L^2}$ and

$$\frac{1}{2}\left\|u\right\|_{L^2} \leq \left\|u + T_V u\right\|_{L^2} \leq 2\left\|u\right\|_{L^2} \tag{8.2.18}$$

if

$$\lambda \geq 2\gamma M_0^1(P;n). \tag{8.2.19}$$

We now suppose that λ is chosen so that this condition is satisfied.

The symbolic calculus and (8.2.16) imply that

$$\big(\partial_t + iA(\partial_x) + T_Q\big)(\mathrm{Id} + T_V) = (\mathrm{Id} + T_V)\big(\partial_t + iA(\partial_x) + T_P\big) + [\partial_t, T_V] + R$$

where R is of order ≤ 0. By definition, $T_V = \sigma_V(t,x,\partial_x)$ where the symbol $\sigma_V \in \Sigma_1^{-1}$. Moreover, since $\partial_t V \in \Gamma_{-1}^{-1}$, Proposition 5.1.13 implies that

$$\sigma_{\partial_t V}(t,\cdot,\cdot) \in \Sigma_0^0$$

and therefore $[\partial_t, T_V] = \sigma_{\partial_t V}$ is of order ≤ 0.

Thus $v = u + T_V u$ satisfies:

$$\partial_t v + iA(\partial_x)v + T_Q v = g, \tag{8.2.20}$$

with

$$\|g(t)\|_{L^2} \le \|(\mathrm{Id} + T_V)f(t)\|_{L^2} + C\|u(t)\|_{L^2}$$

with C as in (8.2.14).

Applying Proposition 8.2.5 to v and equation (8.2.20), and using Gronwall lemma once more, implies the estimate of Proposition 8.2.6. □

Proof of Theorem 8.2.4.

If the B_j satisfy the smoothness conditions of Assumption 8.2.2, then the symbol iB satisfies the assumptions of Proposition 8.2.6. Therefore, there u satisfies energy estimate (8.2.12) with $f = \partial_t u + iA(\partial_x)u + T_{iB}u$. Hence, using the paralinearized equation (8.2.11) and Gronwall's Lemma implies the estimate (8.2.5) of the proposition. □

8.3 Existence, uniqueness and smoothness for linear problems

In this section, we sketch the linear existence theory which can be deduced from the energy estimates. Many proofs are similar to those exposed in the previous chapter and many details are omitted.

8.3.1 L^2 existence

We always assume that the second order system $A(\partial_x)$ satisfies Assumption 8.2.1.

Theorem 8.3.1. *Suppose that Assumption 8.2.2 is satisfied. Then, for $f \in L^1([0,T]; L^2(\mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$ the Cauchy problem for* (8.2.1) *with initial data h has a unique solution $u \in C^0([0,T]; L^2(\mathbb{R}^d))$ which satisfies the energy estimate* (8.2.5).

Sketch of proof. **a)** We use the mollifiers $J_\varepsilon = (1 - \varepsilon\Delta)^{-1}$. The Cauchy problem

$$\partial_t u_\varepsilon + iA(\partial_x)J_\varepsilon u_\varepsilon + B(t,x,\partial_x)J_\varepsilon u_\varepsilon = f, \qquad u_{\varepsilon|t=0} = h \tag{8.3.1}$$

has a unique solution $u_\varepsilon \in C^1([0,T]; L^2(\mathbb{R}^d)$, since the operators $iA(\partial_x)J_\varepsilon$ and $B(t,x,\partial_x)J_\varepsilon$ are bounded in L^2.

The proof of the L^2 energy estimates extends to the equation above, implying that the u_ε are uniformly in $C^0([0,T];L^2(\mathbb{R}^d))$. The equation implies that the $\partial_t u_\varepsilon - f$ are uniformly bounded in $C^0([0,T];H^{-2}(\mathbb{R}^d))$.

Therefore, by Ascoli-Arzela theorem, there is a subsequence, still denoted bu u_ε, which converges in $C^0([0,T];L^2_w(\mathbb{R}^d))$, where L^2_w denotes the space L^2 equipped with the weak topology. There is no difficulty to pass to the limit in the equation, and the limit $u \in C^0([0,T];L^2_w(\mathbb{R}^d)) \subset L^2([0,T]\times\mathbb{R}^d)$ is a (weak) solution of the Cauchy problem

$$Lu = f, \qquad u_{|t=0} = h.$$

b) Repeating the proof of Lemma 7.1.13, one shows that

$$LJ_\varepsilon u \ \to\ f \quad \text{in} \quad L^1([0,T];L^2(\mathbb{R}^d)).$$

Indeed, the commutator $[L, J_\varepsilon]$ reduces to $[B(t,x,\partial_x), J_\varepsilon]$ which can be treated exactly as in the proof of the above mentioned lemma.

Therefore, the energy estimates applied to $J_\varepsilon u - J_{\varepsilon'} u$ imply that the $J_\varepsilon u$ form a Cauchy family in $C^0([0,T];L^2)$. Thus the limit u belongs $L^2([0,T]\times\mathbb{R}^d)$. Moreover, passing to the limit in the energy estimates applied to $J_\varepsilon u$, we see that u also satisfies these estimates. The uniqueness follows. $\square$

8.3.2 H^s existence

Assumption 8.3.2. *The coefficients $B_j(t,x)$ satisfy $\nabla_x B_j \in L^\infty([0,T], H^{s-1}(\mathbb{R}^d))$.*

When this condition is satisfied let

$$N_s = \sum_j \sup_{t\in[0,T]} \big\|\nabla_x B_j(t,\cdot)\big\|_{H^{s-1}(\mathbb{R}^d)} \tag{8.3.2}$$

Theorem 8.3.3. *Suppose that Assumptions 8.2.2 and 8.3.2 with $s > \frac{d}{2}+1$ are satisfied. Then for $f \in L^1([0,T];H^s(\mathbb{R}^d)$ and $h \in H^s(\mathbb{R}^d)$ the Cauchy problem for* (8.2.1) *with initial data h has a unique solution $u \in C^0([0,T];H^s(\mathbb{R}^d))$ which satisfies the energy estimate*

$$\big\|u(t)\big\|_{H^s} \le C_0 e^{tC_1}\Big(\big\|u(0)\big\|_{H^s} + \int_0^t \big\|f(t')\big\|_{H^s} dt'\Big) \tag{8.3.3}$$

where the constants C_0 depends only on K_0 and and C_1 depends on (K_1, N_s), with K_0 and K_1 given by (8.2.6). (8.2.7).

Sketch of proof. **a)** The key point is to obtain H^s energy estimates. If u solves the equation $Lu = f$, then the para-linearization Proposition 5.2.2 implies that

$$\partial_t u + iA(\partial_x) + T_{iB} = f' \tag{8.3.4}$$

with

$$\big\|f(t) - f'(t)\big\|_{H^s} \leq CN_s \big\|u(t)\big\|_{H^s}. \tag{8.3.5}$$

Conjugating the equation (8.3.4) by $(1 - \Delta_x)^{s/2}$ and using the L^2 energy estimate for $(1 - \Delta_x)^{s/2}u$, implies that

$$\big\|u(t)\big\|_{H^s} \leq C_0 e^{tC_1}\Big(\big\|u(0)\big\|_{H^s} + \int_0^t \big\|f'(t')\big\|_{H^s} dt'\Big) \tag{8.3.6}$$

with constants C_0 and C_1 as in (8.2.5) or (**??**). Together with (8.3.5), this implies that $u \in C^0([0, T]; H^{s+2} \cap C^1([0, T]; H^s)$ satisfy the energy estimate (8.3.3).

b) Solutions $u \in C^0([0, T]; H^s_w)$ are constructed using the approximate equations (8.3.1). As in Proposition (7.2.7), commuting L with J_ε, one shows that $u \in C^0([0, T]; H^s(\mathbb{R}^d))$ and satisfies the energy estimates (8.3.3). □

8.4 Nonlinear problems

8.4.1 Systems with quasilinear first order part

We consider a $N \times N$ nonlinear system

$$\partial_t u + iA(\partial_x)u + B(u, \partial_x)u = F(u) \tag{8.4.1}$$

with $A(\partial_x) = \sum A_{j,k}\partial_{x_j}\partial_{x_k}$ satisfying Assumption 8.2.1 and

$$B(u, \partial_x) = \sum B_j(u)\partial_{x_j}. \tag{8.4.2}$$

The matrices $B_j(u)$ and $F(u)$ are supposed to be smooth functions of their argument $u \in \mathbb{R}^N$, with $F(0) = 0$. For simplicity, we suppose that they do not depend on the space time variables (t, x) and leave this extension to the reader.

We still denote by $\Pi_j(\xi)$ the self-adjoint eigenprojectors of $A(\xi)$.

Assumption 8.4.1. [Symmetry of the diagonal blocks.] *For all j and u, the matrix $\Pi_j(\xi)B(u, \xi)\Pi_j(\xi)$ is self adjoint.*

Theorem 8.4.2. *Suppose that Assumptions 8.2.1 and 8.4.1 are satsified. Then, for $s > \frac{d}{2} + 1$ and $h \in H^s(\mathbb{R}^d)$, there is $T > 0$ such that the Cauchy problem for* (8.4.1) *with initial data h has a unique solution $u \in C^0([0,T]; H^s(\mathbb{R}^d))$.*

We solve the equation (8.4.1) by Picard's iteration, and consider the iterative scheme

$$\partial_t u_{n+1} + iA(\partial_x)u_{n+1} + B(u_n, \partial_x)u_{n+1} = F(u_n), \qquad u_{|t=0} = h, \tag{8.4.3}$$

starting from $u_0(t,x) = h(x)$.

Lemma 8.4.3. *The u_n are defined for all n and*

$$u_n \in C^0([0,T]; H^s(\mathbb{R}^d)), \qquad \partial_t u_n \in C^0([0,T]; H^{s-2}(\mathbb{R}^d)). \tag{8.4.4}$$

Proof. This is true for u_0. Suppose that u_n satisfies (8.4.4). Therefore, u_n belongs to $W^{1,\infty}$ as well as the coefficients $B_j(u_n(t,x))$. Moreover, applying Theorem 5.2.6 to $B_j(u) - B_j(0)$, implies that the $B_j(u_n)$ also satisfy the condition $\nabla_x B_j(u_n) \in C^0([0,T], H^{s-1})$.

Assumption 8.4.1 and the condition $s > \frac{d}{2} + 1$ imply that

$$\partial_t B_j(u_n) = (\nabla_u B_j)(u_n)\partial_t u_n \in C^0([0,T]; H^{s-2}) \subset C^0([0,T], W^{-1,\infty}).$$

Moreover, $\Pi_k(\xi)B(u_n(t,x),\xi)\Pi_k(\xi) = 0$ implying that the linear equation (8.4.3) satisfies the Assumption 8.2.2.

Therefore Theorem 8.3.3 can be applied, and (8.4.3) has a unique solution $u_{n+1} \in C^0([0,T]; H^s)$. The equation implies that $\partial_t u_{n+1} \in C^0([0,T]; H^{s-2}(\mathbb{R}^d))$. Thus the lemma follows by induction. □

Lemma 8.4.4. *There is $T' \in]0,T]$, such that the sequences u_n and $\partial_t u_n$ are bounded in $C^0([0,T']; H^s(\mathbb{R}^d))$ and in $C^0([0,T']; H^{s-2}(\mathbb{R}^d))$ respectively.*

Proof. The proof is similar to the proof of Lemma 7.3.6 One proves by induction that there are $T' > 0$ and constants m, R and R_1 such that

$$\|u_n\|_{L^\infty([0,T']\times\mathbb{R}^d)} \le m, \tag{8.4.5}$$

$$\sup_{t\in[0,T']} \|u_n(t)\|_{H^s(\mathbb{R}^d)} \le R, \tag{8.4.6}$$

$$\sup_{t\in[0,T']} \|\partial_t u_n(t)\|_{H^{s-2}(\mathbb{R}^d)} \le R_1. \tag{8.4.7}$$

Assume that u_n satisfies these estimates. Then, the energy estimate (8.3.3) applied to the linear problem (8.4.3) yields and estimate of the form

$$\big\|u_{n+1}(t)\big\|_{H^s} \le Ce^{Kt}\big\|h\big\|_{H^s} + tKe^{Kt} \tag{8.4.8}$$

where $C = C(m)$ depends only on m and $K = K(m, R, R_1)$ depends only on the bounds m, R and R_1. The H^s norm of $F(u_n)$ is also estimated by $K(R)$, and finally, we see that there are constants $C = C(m)$ and $K = K(m, R, R_1)$ such that Moreover, the equation implies that there is a function $D(R)$

$$\big\|\partial_t u_{n+1}(t)\big\|_{H^{s-2}} \le D(m, R)\big(1 + \big\|u_{n+1}(t)\big\|_{H^s}\big). \tag{8.4.9}$$

We choose successively

$$m > \big\|h\big\|_{L^\infty(\mathbb{R}^d)} \tag{8.4.10}$$

$$R > C(m)\big\|h\big\|_{H^s(\mathbb{R}^d)}, \tag{8.4.11}$$

$$R_1 > D(m, R)(1 + R). \tag{8.4.12}$$

Therefore, if T' is small enough,

$$C(m)e^{T'K(m,R,R_1)}\big(\big\|h\big\|_{H^s(\mathbb{R}^d)} + T'K(m, R, R_1)\big) \le R, \tag{8.4.13}$$

and the energy estimate imply that u_{n+1} satisfies (8.4.6) and (8.4.7).

Here is the slight modification with respect to the proof of Lemma 7.3.6: Since $s > \frac{d}{2} + 1$, the bound (8.4.7) for $\partial_t u_n$ in $C^0(H^{s_2})$ does not give control of $\partial_t u_n$ in L^∞ and thus (7.3.15) is not guaranteed any more. Instead, we note that (8.4.6) and (8.4.7) imply that $u_n \in C^{\frac{1}{2}}([0, T']; H^{s-1}(\mathbb{R}^d)$ with norm bounded by $\gamma(R + R_1)$ and therefore

$$\big\|u_{n+1}(t) - h\big\|_{L^\infty} \le \gamma\big\|u_{n+1}(t) - h\big\|_{H^{s-1}} \le \sqrt{t}\gamma(R + R_1). \tag{8.4.14}$$

Therefore, if T' is small enough and $t \le T'$, the right hand side is $\le m$, implying that u_{n+1} satisfies (8.4.5). □

Lemma 8.4.5. *The sequence u_n is a Cauchy sequence in $C^0([0, T']; L^2(\mathbb{R}^d))$.*

Proof. It is identical to the proof of Lemma 7.3.7: one writes the equation for $v_n := u_{n+1} - u_n$ and use the L^2 energy estimates, which imply that the series $\sum v_n$ converges in $C^0([0, T']; L^2(\mathbb{R}^d))$. □

Proof of Theorem 8.4.2.

The uniform bounds and the convergence in $C^0([0,T'];L^2(\mathbb{R}^d))$ imply that the sequence u_n converges in $C^0([0,T'];H^{s'}(\mathbb{R}^d))$ for all $s'<s$. Similarly, $\partial_t u_n$ converges in $C^0([0,T'];H^{s'-2}(\mathbb{R}^d))$ This implies that the limit u belongs to $C^0([0,T'];H^{s'}(\mathbb{R}^d))$ and to $L^\infty([0,T];H^s)$, with $\partial_t u$ belonging to $C^0([0,T'];H^{s'-2}(\mathbb{R}^d))$ and $L^\infty([0;T],H^{s-2})$. Moreover, u is solution of the equation.

The next step consists in considering (8.4.1) as a linear equation for $u\in L^2([0,T'];H^s)$ with coefficients B_j in $L^\infty([0,T'],H^s)$. Theorem 8.3.3 implies that $u\in C^0([0,T];H^s)$. $\square$

8.4.2 Examples and applications

Systems with diagonal second order term

Suppose that A is block diagonal

$$A=\begin{pmatrix}\lambda_1(\xi)\mathrm{Id}_{N_1} & 0 & \dots & 0\\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & \lambda_p\mathrm{Id}_{N_p}\end{pmatrix}=\mathrm{diag}\{\lambda_j\mathrm{Id}_{N_j}\}\tag{8.4.15}$$

with $\lambda_j(\xi)$ second order homogeneous polynomial of degree two with real coefficients.

Assumption 8.2.1 is satisfied if the $\lambda_j(\xi)\neq\lambda_k(\xi)$ for $j\neq k$ and $\xi\neq 0$.

In the block decomposition (8.4.15) write

$$B_j(u)=\begin{pmatrix}B_j^{1,1}(u) & \dots & B_j^{1,p}(u)\\ & \ddots & \\ B_j^{p,1}(u) & \dots & B_j^{p,p}(u)\end{pmatrix}=\left(B_j^{k,l}(u)\right).\tag{8.4.16}$$

Then Assumption 8.4.1 is satisfied when the diagonal blocks $B_j^{k,k}(u)$ are self adjoint matrices for all u, in particular when they vanish meaning that there are no self-interaction between the components u_k.

This applies in particular to the systems (8.1.1) mentioned in the introduction and Theorems 8.1.1 is a corollary of Theroem 8.4.2.

Systems involving u and $\overline{u}$

For applications, it is interesting to make explicit the result when the first order part also depends on $\overline{u}$. Consider the system

$$\partial_t u + iA(\partial_x)u + B(u,\partial_x)u + C(u,\partial_x)\overline{u} = 0 \tag{8.4.17}$$

where $A(\xi) = \operatorname{diag}\{\lambda_k Id_{N_k}\}$ as in (8.4.15). Introducing $v = \overline{u}$ as a variable and setting $U = {}^t(u,v)$, the equation reads:

$$\partial_t U + i\mathcal{A}(\partial_x)U + \mathcal{B}(u,\partial_x)U = 0 \tag{8.4.18}$$

with

$$\mathcal{A} = \begin{pmatrix} A(\partial_x) & 0 \\ 0 & -A(\partial_x) \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} B & C \\ \overline{C} & \overline{B} \end{pmatrix}. \tag{8.4.19}$$

In this context, the Assumption 8.2.1 for $\mathcal{A}$ follows from

Assumption 8.4.6. *For all $\xi \in \mathbb{R}^n \backslash \{0\}$, $A(\xi)$ is self-adjoint with eigenvalues $\lambda_j(\xi)$ of constant multiplicity and $\lambda_j(\xi) + \lambda_k(\xi) \neq 0$ for all j and k.*

Note that for $j = k$ this implies that the quadratic form $\lambda_j(\xi)$ does not vanish for $\xi \neq 0$, and therefore is definite positive or definite negative. This rules out interesting cases of "nonelliptic" Schrödinger equations, which are considered for instance in [KPV].

The diagonal term of $\mathcal{B}_j(u)$ are $B_j^{k,k}(u)$ and $\overline{B_j^{k,k}(u)}$. Therefore, Assumption 8.4.1 for $\mathcal{B}$ reads

Assumption 8.4.7. *For all k and j, $\operatorname{Im} B_j^{k,k}(u) = 0$.*

Systems with fully nonlinear first order part

Next we briefly discuss the case of equations with fully nonlinear right hand side:

$$\partial_t u + iA(\partial_x)u + F(u,\partial_x u) = 0, \tag{8.4.20}$$

where $F(t,x,u,v_1,\ldots,v_d)$ is a smooth function of $(t,x,\operatorname{Re} u,\operatorname{Im} u)$ and of $(\operatorname{Re} v_1,\ldots,\operatorname{Im} v_d)$. For simplicity, we assume that $A(\xi) = \operatorname{diag}\{\lambda_k Id_{N_k}\}$ as in (8.4.15).

All the analysis relies on a para-linearization of the first order term. Following the para-linearization Theorem 5.2.4, the analogues of the symbols $B(u,\xi)$ and $C(u,\xi)$ in (8.4.17) are

$$B(u,v,\xi)=\sum_j \xi_j \nabla_{v_j} F(u,v) \tag{8.4.21}$$

$$C(u,v,\xi)=\sum_j \xi_j \nabla_{\overline{v}_j} F(u,v) \tag{8.4.22}$$

with

$$\nabla_{v_j}=\frac{1}{2}\nabla_{\mathrm{Re}\, v_j}-\frac{i}{2}\nabla_{\mathrm{Im}\, v_j}, \qquad \nabla_{\overline{v}_j}=\frac{1}{2}\nabla_{\mathrm{Re}\, v_j}+\frac{i}{2}\nabla_{\mathrm{Im}\, v_j}.$$

Let $B_j(u,v)=\nabla_{v_j}F(u,v)$, and let us denote by $B_j^{k,k}(u,v)$ its k-th block diagonal part in the block decomposition of A. If $(f^1,\dots,f^p)$ denote the block components of a vector f, there holds

$$B_j^{k,k}(u,v)=\nabla_{v_j^k}F^k(u,v). \tag{8.4.23}$$

The analogue of Assumption 8.4.7 is

Assumption 8.4.8. *For all k and j, $\mathrm{Im}\, B_j^{k,k}(u,v)=0$.*

Using the para-linearization Theorem 5.2.4 and the energy estimates of Sections 2 and 3 for the para-linear equations, one obtains *a priori* estimates for the solutions of (8.4.20), provided that the smoothness of the coefficients remains sufficient. Alternately, one can differentiate the equation and use that the $v_j=\partial_{x_j}u$ satisfy

$$\partial_t v_j+iA(\partial_x)v_j+B(u,v,\partial_x)v_j+C(u,v,\partial_x)\overline{v}_j+\nabla_u F(u,v)v_j=0, \tag{8.4.24}$$

The Sobolev *a priori* estimates are the key point to prove the existence os solutions. For instance, one can prove the following result:

Theorem 8.4.9. *Suppose that Assumptions 8.4.6 and 8.4.8 [resp 8.4.9] are satsified. Then, for $s>\frac{d}{2}+2$ [resp. $s>\frac{d}{2}+3$] and $h\in H^s(\mathbb{R}^d)$, there is $T>0$ such that the Cauchy problem for (8.4.20) with initial data h has a unique solution $u\in C^0([0,T];H^s(\mathbb{R}^d))$.*

Bibliography

[Blo] N. Bloembergen, *Nonlinear Optics*, W.A. Benjamin Inc., New York, 1965

[Bon] J.M.Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann.Sc.E.N.S. Paris, 14 (1981) pp 209-246.

[Boy] R. Boyd, *Nonlinear Optics*, Academic Press, 1992

[Ch-Pi] J.Chazarain, A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Bordas, Paris, 1981.

[Ch] H.Chihara, *Local existence for semilinear Schrödinger equations*, Math.Japonica 42 (1995) pp 35-52.

[CM] R.Coifman and Y.Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57 (1978).

[CC] M. Colin, T. Colin, *On a quasilinear Zakharov System describing laser-plasma interactions*, Differential and Integral Equations, 17 (2004) pp 297-330.

[CCM] M. Colin, T. Colin, M.Métivier, *Nonlinear models for laser-plasma interaction*, Séminaire X-EDP, École Polytechnique, (2007).

[Co-Fr] R.Courant, K.O.Friedrichs, *Supersonic flow and shock waves,* Wiley-Interscience, New York, 1948.

[Co-Hi] R.Courant, D.Hilbert, *Methods of Mathematical Physics,* John Wiley & Sons, New York, 1962.

[Fr1] K.0.Friedrichs, *Symmetric hyperbolic linear differential equations,* Comm. Pure Appl.Math., 7 (1954), pp 345-392.

[Fr2] K.0.Friedrichs, *Symmetric positive linear differential equations,* Comm. Pure Appl.Math., 11 (1958), pp 333-418.

[Fr3] K.0.Friedrichs *Nonlinear hyperbolic differential equations for functions of two independent variables,* Amer. J. Math., 70 (1948) pp 555-589.

[Gå] L. Gårding, *Problèmes de Cauchy pour des systèmes quasi-linéaires d'ordre un, strictement hyperboliques,* dans *Les EDP,* Colloques Internationaux du CNRS, vol 117, (1963), pp 33-40.

[Hör] L.Hörmander, *Lectures on Nonlinear Hyprbolic Differential Equations*, Mathématiques et Applications 26, Sringer Verlag, 1997.

[KPV] E.Kenig, G.Ponce, L.Vega, *Smoothing effects and Local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math. 134 (1998) pp 489-545.

[La] P.Lax, *Hyperbolic systems of conservation laws*, Comm. Pure Appl. Math., 10 (1957), pp 537-566.

[Ma1] A.Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc., n^o 275, 1983.

[Ma2] A.Majda, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc., n^o 281, 1983.

[Ma3] A.Majda, *Compressible fluid flows and systems of conservation laws,* Applied Math.Sc., 53, Springer Verlag, 1953.

[Mé1] G. Métivier, *Stability of multidimensional shocks.* Advances in the theory of shock waves, 25–103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[Mé2] G. Métivier, *Small viscosity and boundary layer methods. Theory, stability analysis, and applications.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2004.

[MZ] G. Métivier, K.Zumbrun, *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs of the AMS, number 826 (2005).

[Mey] Y.Meyer, *Remarques sur un théorème de J.M.Bony*, Supplemento al Rendiconti der Circolo Matematico di Palermo, Serie II, No1, 1981.

[Miz] S. Mizohata, *On the Cauchy problem*, Notes and Reports in Mathematics in Science an Engineering, 3, Academic Press, 1985.

[NM] A. Newell, J. Moloney, *Nonlinear Optics* Addison-Wesley, Reading Mass., 1992.

[Ste] E.Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[Tay] M.Taylor. *Partial Differential Equations*III, Applied Mathematical Sciences 117, Springer, 1996.